THEORY OF ELASTICITY ☆ LANDAU AND LIFSHITZ

Second Edition

Course of Theoretical Physics Volume 7

THEORY OF ELASTICITY

L. D. LANDAU AND E. M. LIFSHITZ

OTHER TITLES IN THE SERIES

Vol. 1. MECHANICS

Vol. 2. THE CLASSICAL THEORY OF FIELDS

Vol. 3. QUANTUM MECHANICS—NON-RELATIVISTIC THEORY

Vol. 4. RELATIVISTIC QUANTUM THEORY

Vol. 5. STATISTICAL PHYSICS

Vol. 6. FLUID MECHANICS

Vol. 8. ELECTRODYNAMICS OF CONTINUOUS MEDIA

Vol. 9. PHYSICAL KINETICS

THEORY OF ELASTICITY

by

L. D. LANDAU AND E. M. LIFSHITZ

INSTITUTE OF PHYSICAL PROBLEMS, U.S.S.R. ACADEMY OF SCIENCES

Volume 7 of *Course of Theoretical Physics*

Translated from the Russian by

J. B. SYKES AND W. H. REID

SECOND ENGLISH EDITION, REVISED AND ENLARGED



PERGAMON PRESS

Oxford · New York · Toronto

Sydney · Paris · Braunschweig

Pergamon Press Offices:

U. K.	Pergamon Press Ltd., Headington Hill Hall, Oxford OX3 0BW, England
U. S. A.	Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523, U.S.A.
CANADA	Pergamon of Canada, Ltd., 207 Queen's Quay West, Toronto 1, Canada
AUSTRALIA	Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street, Rushcutters Bay, N.S.W. 2011, Australia
FRANCE	Pergamon Press SARL, 24 rue des Ecoles, 75240 Paris, Cedex 05, France
WEST GERMANY	Pergamon Press GMbH, 3300 Braunschweig, Postfach 2923, Burgplatz 1, West Germany

First English edition 1959

Second (revised) edition 1970

Reprinted 1975

Library of Congress Catalog Card No. 77-91701

Reproduced Photolitho in Great Britain by J W Arrowsmith Ltd., Bristol

ISBN 0 08 006465 5

CONTENTS

PREFACE TO THE FIRST ENGLISH EDITION

THE present volume of our *Theoretical Physics* deals with the theory of elasticity.

Being written by physicists, and primarily for physicists, it naturally includes not only the ordinary theory of the deformation of solids, but also some topics not usually found in textbooks on the subject, such as thermal conduction and viscosity in solids, and various problems in the theory of elastic vibrations and waves. On the other hand, we have discussed only very briefly certain special matters, such as complex mathematical methods in the theory of elasticity and the theory of shells, which are outside the scope of this book.

Our thanks are due to Dr. Sykes and Dr. Reid for their excellent translation of the book.

Moscow

L. D. LANDAU
E. M. LIFSHITZ

PREFACE TO THE SECOND ENGLISH EDITION

AS WELL as some minor corrections and additions, a chapter on the macroscopic theory of dislocations has been added in this edition. The chapter has been written jointly by myself and A. M. Kosevich.

A number of useful comments have been made by G. I. Barenblatt, V. L. Ginzburg, M. A. Isakovich, I. M. Lifshitz and I. M. Shmushkevich for the Russian edition, while the vigilance of Dr. Sykes and Dr. Reid has made it possible to eliminate some further errors from the English translation.

I should like to express here my sincere gratitude to all the above-named.

Moscow

E. M. LIFSHITZ

NOTATION

ρ density of matter

$\mathbf{u}$ displacement vector

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \text{ strain tensor}$$

σ_{ik} stress tensor

K modulus of compression

μ modulus of rigidity

E Young's modulus

σ Poisson's ratio

c_l longitudinal velocity of sound

c_t transverse velocity of sound

c_l and c_t are expressed in terms of K, μ or of E, σ by formulae given in §22.

The quantities K, μ, E and σ are related by

$$E = 9K\mu/(3K+\mu)$$
$$\sigma = (3K-2\mu)/2(3K+\mu)$$
$$K = E/3(1-2\sigma)$$
$$\mu = E/2(1+\sigma)$$

CHAPTER I

FUNDAMENTAL EQUATIONS

§1. The strain tensor

THE mechanics of solid bodies, regarded as continuous media, forms the content of the *theory of elasticity*.†

Under the action of applied forces, solid bodies exhibit deformation to some extent, i.e. they change in shape and volume. The deformation of a body is described mathematically in the following way. The position of any point in the body is defined by its radius vector $\mathbf{r}$ (with components $x_1 = x$, $x_2 = y$, $x_3 = z$) in some co-ordinate system. When the body is deformed, every point in it is in general displaced. Let us consider some particular point; let its radius vector before the deformation be $\mathbf{r}$, and after the deformation have a different value $\mathbf{r}'$ (with components x'_i). The displacement of this point due to the deformation is then given by the vector $\mathbf{r}' - \mathbf{r}$, which we shall denote by $\mathbf{u}$:

$$u_i = x'_i - x_i. \tag{1.1}$$

The vector $\mathbf{u}$ is called the *displacement vector*. The co-ordinates x'_i of the displaced point are, of course, functions of the co-ordinates x_i of the point before displacement. The displacement vector u_i is therefore also a function of the co-ordinates x_i. If the vector $\mathbf{u}$ is given as a function of x_i, the deformation of the body is entirely determined.

When a body is deformed, the distances between its points change. Let us consider two points very close together. If the radius vector joining them before the deformation is $\mathrm{d}x_i$, the radius vector joining the same two points in the deformed body is $\mathrm{d}x'_i = \mathrm{d}x_i + \mathrm{d}u_i$. The distance between the points is $\mathrm{d}l = \sqrt{(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)}$ before the deformation, and $\mathrm{d}l' = \sqrt{(\mathrm{d}x'^2_1 + \mathrm{d}x'^2_2 + \mathrm{d}x'^2_3)}$ after it. Using the general summation rule,‡ we can write $\mathrm{d}l^2 = \mathrm{d}x_i^2$, $\mathrm{d}l'^2 = \mathrm{d}x'^2_i = (\mathrm{d}x_i + \mathrm{d}u_i)^2$. Substituting $\mathrm{d}u_i = (\partial u_i/\partial x_k)\mathrm{d}x_k$, we can write

$$\mathrm{d}l'^2 = \mathrm{d}l^2 + 2\frac{\partial u_i}{\partial x_k}\mathrm{d}x_i\,\mathrm{d}x_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\mathrm{d}x_k\,\mathrm{d}x_l.$$

Since the summation is taken over both suffixes i and k in the second term on the right, we can put $(\partial u_i/\partial x_k)\mathrm{d}x_i\mathrm{d}x_k = (\partial u_k/\partial x_i)\mathrm{d}x_i\,\mathrm{d}x_k$. In the third

† The basic equations of elasticity theory were established in the 1820's by CAUCHY and by POISSON.

‡ In accordance with the usual rule, we omit the sign of summation over vector and tensor suffixes. Summation over the values 1, 2, 3 is understood with respect to all suffixes which appear twice in a given term.

1*

term, we interchange the suffixes i and l. Then dl'^2 takes the final form

$$dl'^2 = dl^2 + 2u_{ik}\, dx_i\, dx_k, \tag{1.2}$$

where the tensor u_{ik} is defined as

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right). \tag{1.3}$$

These expressions give the change in an element of length when the body is deformed.

The tensor u_{ik} is called the *strain tensor*. We see from its definition that it is symmetrical, i.e.

$$u_{ik} = u_{ki}. \tag{1.4}$$

This result has been obtained by writing the term $2(\partial u_i/\partial x_k)dx_i\, dx_k$ in dl'^2 in the explicitly symmetrical form

$$\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) dx_i\, dx_k.$$

Like any symmetrical tensor, u_{ik} can be *diagonalised* at any given point. This means that, at any given point, we can choose co-ordinate axes (the *principal axes* of the tensor) in such a way that only the diagonal components u_{11}, u_{22}, u_{33} of the tensor u_{ik} are different from zero. These components, the *principal values* of the strain tensor, will be denoted by $u^{(1)}$, $u^{(2)}$, $u^{(3)}$. It should be remembered, of course, that, if the tensor u_{ik} is diagonalised at any point in the body, it will not in general be diagonal at any other point.

If the strain tensor is diagonalised at a given point, the element of length (1.2) near it becomes

$$\begin{aligned} dl'^2 &= (\delta_{ik} + 2u_{ik})\, dx_i\, dx_k \\ &= (1+2u^{(1)})\, dx_1^2 + (1+2u^{(2)})\, dx_2^2 + (1+2u^{(3)})\, dx_3^2. \end{aligned}$$

We see that the expression is the sum of three independent terms. This means that the strain in any volume element may be regarded as composed of independent strains in three mutually perpendicular directions, namely those of the principal axes of the strain tensor. Each of these strains is a simple extension (or compression) in the corresponding direction: the length dx_1 along the first principal axis becomes $dx'_1 = \sqrt{(1+2u^{(1)})}\, dx_1$, and similarly for the other two axes. The quantity $\sqrt{(1+2u^{(i)})}-1$ is consequently equal to the relative extension $(dx'_i - dx_i)/dx_i$ along the ith principal axis.

In almost all cases occurring in practice, the strains are small. This means that the change in any distance in the body is small compared with the distance itself. In other words, the relative extensions are small compared with unity. In what follows we shall suppose that all strains are small.

If a body is subjected to a small deformation, all the components of the strain tensor are small, since they give, as we have seen, the relative changes in lengths in the body. The displacement vector u_i, however, may sometimes be large, even for small strains. For example, let us consider a long thin rod. Even for a large deflection, in which the ends of the rod move

a considerable distance, the extensions and compressions in the rod itself will be small.

Except in such special cases,† the displacement vector for a small deformation is itself small. For it is evident that a three-dimensional body (i.e. one whose dimension in no direction is small) cannot be deformed in such a way that parts of it move a considerable distance without the occurrence of considerable extensions and compressions in the body.

Thin rods will be discussed in Chapter II. In other cases u_i is small for small deformations, and we can therefore neglect the last term in the general expression (1.3), as being of the second order of smallness. Thus, for small deformations, the strain tensor is given by

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{1.5}$$

The relative extensions of the elements of length along the principal axes of the strain tensor (at a given point) are, to within higher-order quantities, $\sqrt{(1+2u^{(i)})}-1 \approx u^{(i)}$, i.e. they are the principal values of the tensor u_{ik}.

Let us consider an infinitesimal volume element $\mathrm{d}V$, and find its volume $\mathrm{d}V'$ after the deformation. To do so, we take the principal axes of the strain tensor, at the point considered, as the co-ordinate axes. Then the elements of length $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ along these axes become, after the deformation, $\mathrm{d}x'_1 = (1+u^{(1)})\,\mathrm{d}x_1$, etc. The volume $\mathrm{d}V$ is the product $\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$, while $\mathrm{d}V'$ is $\mathrm{d}x'_1\,\mathrm{d}x'_2\,\mathrm{d}x'_3$. Thus $\mathrm{d}V' = \mathrm{d}V(1+u^{(1)})(1+u^{(2)})(1+u^{(3)})$. Neglecting higher order terms, we therefore have $\mathrm{d}V' = \mathrm{d}V(1+u^{(1)}+u^{(2)}+u^{(3)})$. The sum $u^{(1)}+u^{(2)}+u^{(3)}$ of the principal values of a tensor is well known to be invariant, and is equal to the sum of the diagonal components $u_{ii} = u_{11}+u_{22}+u_{33}$ in any co-ordinate system. Thus

$$\mathrm{d}V' = \mathrm{d}V(1+u_{ii}). \tag{1.6}$$

We see that the sum of the diagonal components of the strain tensor is the relative volume change $(\mathrm{d}V'-\mathrm{d}V)/\mathrm{d}V$.

It is often convenient to use the components of the strain tensor in spherical or cylindrical co-ordinates. We give here, for reference, the corresponding formulae, which express the components in terms of the derivatives of the components of the displacement vector in the same co-ordinates. In spherical co-ordinates r, θ, ϕ, we have

$$\left.\begin{aligned}
u_{rr} &= \frac{\partial u_r}{\partial r}, \quad u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}, \quad u_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r},\\
2u_{\theta\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta}-u_\phi\cot\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}, \quad 2u_{r\theta} = \frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta},\\
2u_{\phi r} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}.
\end{aligned}\right\} \tag{1.7}$$

† Which include, besides deformations of thin rods, those of thin plates to form cylindrical surfaces

In cylindrical co-ordinates r, ϕ, z,

$$\left.\begin{aligned} u_{rr} &= \frac{\partial u_r}{\partial r}, \quad u_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}, \quad u_{zz} = \frac{\partial u_z}{\partial z}, \\ 2u_{\phi z} &= \frac{1}{r}\frac{\partial u_z}{\partial \phi}+\frac{\partial u_\phi}{\partial z}, \quad 2u_{rz} = \frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}, \\ 2u_{r\phi} &= \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \phi}. \end{aligned}\right\} \qquad (1.8)$$

§2. The stress tensor

In a body that is not deformed, the arrangement of the molecules corresponds to a state of thermal equilibrium. All parts of the body are in mechanical equilibrium. This means that, if some portion of the body is considered, the resultant of the forces on that portion is zero.

When a deformation occurs, the arrangement of the molecules is changed, and the body ceases to be in its original state of equilibrium. Forces therefore arise which tend to return the body to equilibrium. These internal forces which occur when a body is deformed are called *internal stresses*. If no deformation occurs, there are no internal stresses.

The internal stresses are due to molecular forces, i.e. the forces of interaction between the molecules. An important fact in the theory of elasticity is that the molecular forces have a very short range of action. Their effect extends only to the neighbourhood of the molecule exerting them, over a distance of the same order as that between the molecules, whereas in the theory of elasticity, which is a macroscopic theory, the only distances considered are those large compared with the distances between the molecules. The range of action of the molecular forces should therefore be taken as zero in the theory of elasticity. We can say that the forces which cause the internal stresses are, as regards the theory of elasticity, "near-action" forces, which act from any point only to neighbouring points. Hence it follows that the forces exerted on any part of the body by surrounding parts act only on the surface of that part.

The following reservation should be made here. The above asserioon is not valid in cases where the deformation of the body results in macroscopic electric fields in it (pyroelectric and piezoelectric bodies). We shall not discuss such bodies in this book, however.

Let us consider the total force on some portion of the body. Firstly, this total force is equal to the sum of all the forces on all the volume elements in that portion of the body, i.e. it can be written as the volume integral $\int \mathbf{F}\,\mathrm{d}V$, where $\mathbf{F}$ is the force per unit volume and $\mathbf{F}\mathrm{d}V$ the force on the volume element $\mathrm{d}V$. Secondly, the forces with which various parts of the portion considered act on one another cannot give anything but zero in the total resultant force, since they cancel by NEWTON's third law. The required total force can therefore be regarded as the sum of the forces exerted on the given portion of the

body by the portions surrounding it. From above, however, these forces act on the surface of that portion, and so the resultant force can be represented as the sum of forces acting on all the surface elements, i.e. as an integral over the surface.

Thus, for any portion of the body, each of the three components $\int F_i \, \mathrm{d}V$ of the resultant of all the internal stresses can be transformed into an integral over the surface. As we know from vector analysis, the integral of a scalar over an arbitrary volume can be transformed into an integral over the surface if the scalar is the divergence of a vector. In the present case we have the integral of a vector, and not of a scalar. Hence the vector F_i must be the divergence of a tensor of rank two, i.e. be of the form

$$F_i = \partial\sigma_{ik}/\partial x_k. \tag{2.1}$$

Then the force on any volume can be written as an integral over the closed surface bounding that volume:†

$$\int F_i \, \mathrm{d}V = \int \frac{\partial \sigma_{ik}}{\partial x_k} \mathrm{d}V = \oint \sigma_{ik} \, \mathrm{d}f_k, \tag{2.2}$$

where $\mathrm{d}f_i$ are the components of the surface element vector $\mathbf{df}$, directed (as usual) along the outward normal.‡

The tensor σ_{ik} is called the *stress tensor*. As we see from (2.2), $\sigma_{ik}\mathrm{d}f_k$ is the ith component of the force on the surface element $\mathbf{df}$. By taking elements of area in the planes of xy, yz, zx, we find that the component σ_{ik} of the stress tensor is the ith component of the force on unit area perpendicular to the x_k-axis. For instance, the force on unit area perpendicular to the x-axis, normal to the area (i.e. along the x-axis) is σ_{xx}, and the tangential forces (along the y and z axes) are σ_{yx} and σ_{zx}.

The following remark should be made concerning the sign of the force $\sigma_{ik}\mathrm{d}f_k$. The surface integral in (2.2) is the force exerted on the volume enclosed by the surface by the surrounding parts of the body. The force which this volume exerts on the surface surrounding it is the same with the opposite sign. Hence, for example, the force exerted by the internal stresses on the surface of the body itself is $-\oint \sigma_{ik}\mathrm{d}f_k$, where the integral is taken over the surface of the body and $\mathbf{df}$ is along the outward normal.

Let us determine the moment of the forces on a portion of the body. The moment of the force $\mathbf{F}$ can be written as an antisymmetrical tensor of rank two, whose components are $F_i x_k - F_k x_i$, where x_i are the co-ordinates of the

† The integral over a closed surface is transformed into one over the volume enclosed by the surface by replacing the surface element $\mathrm{d}f_i$ by the operator $\mathrm{d}V\partial/\partial x_i$.

‡ Strictly speaking, to determine the total force on a deformed portion of the body we should integrate, not over the old co-ordinates x_i, but over the co-ordinates x'_i of the points of the deformed body. The derivatives (2.1) should therefore be taken with respect to x'_i. However, in view of the smallness of the deformation, the derivatives with respect to x_i and x'_i differ only by higher order quantities, and so the derivatives can be taken with respect to the co-ordinates x_i.

point where the force is applied.† Hence the moment of the forces on the volume element dV is $(F_i x_k - F_k x_i)dV$, and the moment of the forces on the whole volume is $M_{ik} = \int(F_i x_k - F_k x_i)dV$. Like the total force on any volume, this moment can be expressed as an integral over the surface bounding the volume. Substituting the expression (2.1) for F_i, we find

$$M_{ik} = \int\left(\frac{\partial \sigma_{il}}{\partial x_l}x_k - \frac{\partial \sigma_{kl}}{\partial x_l}x_i\right) dV$$

$$= \int \frac{\partial(\sigma_{il}x_k - \sigma_{kl}x_i)}{\partial x_l} dV - \int\left(\sigma_{il}\frac{\partial x_k}{\partial x_l} - \sigma_{kl}\frac{\partial x_i}{\partial x_l}\right) dV.$$

In the second term we use the fact that the derivative of a co-ordinate with respect to itself is unity, and with respect to another co-ordinate is zero (since the three co-ordinates are independent variables). Thus $\partial x_k/\partial x_l = \delta_{kl}$, where δ_{kl} is the unit tensor; the multiplication gives $\sigma_{il}\delta_{kl} = \sigma_{ik}$, $\sigma_{kl}\delta_{il} = \sigma_{ki}$. In the first term, the integrand is the divergence of a tensor; the integral can be transformed into one over the surface. The result is $M_{ik} = \oint(\sigma_{il}x_k - \sigma_{kl}x_i)df_l + \int(\sigma_{ki} - \sigma_{ik})dV$. If M_{ik} is to be an integral over the surface only, the second term must vanish identically, i.e. we must have

$$\sigma_{ik} = \sigma_{ki}. \tag{2.3}$$

Thus we reach the important result that the stress tensor is symmetrical. The moment of the forces on a portion of the body can then be written simply as

$$M_{ik} = \int(F_i x_k - F_k x_i)\, dV = \oint(\sigma_{il}x_k - \sigma_{kl}x_i)\, df_l. \tag{2.4}$$

It is easy to find the stress tensor for a body undergoing uniform compression from all sides (*hydrostatic* compression). In this case a pressure of the same magnitude acts on every unit area on the surface of the body, and its direction is along the inward normal. If this pressure is denoted by p, a force $-p\,df_i$ acts on the surface element df_i. This force, in terms of the stress tensor, must be $\sigma_{ik}df_k$. Writing $-p\,df_i = -p\delta_{ik}df_k$, we see that the stress tensor in hydrostatic compression is

$$\sigma_{ik} = -p\delta_{ik}. \tag{2.5}$$

Its non-zero components are simply equal to the pressure.

In the general case of an arbitrary deformation, the non-diagonal components of the stress tensor are also non-zero. This means that not only a normal force but also tangential (shearing) stresses act on each surface element. These latter stresses tend to move the surface elements relative to each other.

---

† The moment of the force **F** is defined as the vector product **F** × **r**, and we know from vector analysis that the components of a vector product form an antisymmetrical tensor of rank two as written here.

In equilibrium the internal stresses in every volume element must balance, i.e. we must have $F_i = 0$. Thus the equations of equilibrium for a deformed body are

$$\partial\sigma_{ik}/\partial x_k = 0. \tag{2.6}$$

If the body is in a gravitational field, the sum $\mathbf{F}+\rho\mathbf{g}$ of the internal stresses and the force of gravity ($\rho\mathbf{g}$ per unit volume) must vanish; ρ is the density† and $\mathbf{g}$ the gravitational acceleration vector, directed vertically downwards. In this case the equations of equilibrium are

$$\partial\sigma_{ik}/\partial x_k + \rho g_i = 0. \tag{2.7}$$

The external forces applied to the surface of the body (which are the usual cause of deformation) appear in the boundary conditions on the equations of equilibrium. Let $\mathbf{P}$ be the external force on unit area of the surface of the body, so that a force $\mathbf{P}\,\mathrm{d}f$ acts on a surface element $\mathrm{d}f$. In equilibrium, this must be balanced by the force $-\sigma_{ik}\,\mathrm{d}f_k$ of the internal stresses acting on that element. Thus we must have $P_i\,\mathrm{d}f - \sigma_{ik}\,\mathrm{d}f_k = 0$. Writing $\mathrm{d}f_k = n_k\,\mathrm{d}f$, where $\mathbf{n}$ is a unit vector along the outward normal to the surface, we find

$$\sigma_{ik}n_k = P_i. \tag{2.8}$$

This is the condition which must be satisfied at every point on the surface of a body in equilibrium.

We shall derive also a formula giving the mean value of the stress tensor in a deformed body. To do so, we multiply equation (2.6) by x_k and integrate over the whole volume:

$$\int \frac{\partial\sigma_{il}}{\partial x_l} x_k\,\mathrm{d}V = \int \frac{\partial(\sigma_{il}x_k)}{\partial x_l}\,\mathrm{d}V - \int \sigma_{il}\frac{\partial x_k}{\partial x_l}\,\mathrm{d}V = 0.$$

The first integral on the right is transformed into a surface integral; in the second integral we put $\partial x_k/\partial x_l = \delta_{kl}$. The result is $\oint \sigma_{il}x_k\,\mathrm{d}f_l - \int \sigma_{ik}\,\mathrm{d}V = 0$. Substituting (2.8) in the first integral, we find $\oint P_i x_k\,\mathrm{d}f = \int \sigma_{ik}\,\mathrm{d}V = V\bar{\sigma}_{ik}$, where V is the volume of the body and $\bar{\sigma}_{ik}$ the mean value of the stress tensor. Since $\sigma_{ik} = \sigma_{ki}$, this formula can be written in the symmetrical form

$$\bar{\sigma}_{ik} = (1/2V)\oint (P_i x_k + P_k x_i)\,\mathrm{d}f. \tag{2.9}$$

Thus the mean value of the stress tensor can be found immediately from the external forces acting on the body, without solving the equations of equilibrium.

---

† Strictly speaking, the density of a body changes when it is deformed. An allowance for this change, however, involves higher-order quantities in the case of small deformations, and is therefore unimportant.

§3. The thermodynamics of deformation

Let us consider some deformed body, and suppose that the deformation is changed in such a way that the displacement vector u_i changes by a small amount δu_i; and let us determine the work done by the internal stresses in this change. Multiplying the force $F_i = \partial\sigma_{ik}/\partial x_k$ by the displacement δu_i and integrating over the volume of the body, we have $\int \delta R \, \mathrm{d}V = \int(\partial\sigma_{ik}/\partial x_k)\delta u_i \, \mathrm{d}V$, where δR denotes the work done by the internal stresses per unit volume. We integrate by parts, obtaining

$$\int \delta R \, \mathrm{d}V = \oint \sigma_{ik}\delta u_i \, \mathrm{d}f_k - \int \sigma_{ik}\frac{\partial \delta u_i}{\partial x_k} \, \mathrm{d}V.$$

By considering an infinite medium which is not deformed at infinity, we make the surface of integration in the first integral tend to infinity; then $\sigma_{ik} = 0$ on the surface, and the integral is zero. The second integral can, by virtue of the symmetry of the tensor σ_{ik}, be written

$$\begin{aligned}\int \delta R \, \mathrm{d}V &= -\frac{1}{2}\int \sigma_{ik}\left(\frac{\partial \delta u_i}{\partial x_k} + \frac{\partial \delta u_k}{\partial x_i}\right) \mathrm{d}V \\ &= -\frac{1}{2}\int \sigma_{ik}\delta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \mathrm{d}V \\ &= -\int \sigma_{ik}\delta u_{ik} \, \mathrm{d}V.\end{aligned}$$

Thus we find

$$\delta R = -\sigma_{ik}\delta u_{ik}. \tag{3.1}$$

This formula gives the work δR in terms of the change in the strain tensor.

If the deformation of the body is fairly small, it returns to its original undeformed state when the external forces causing the deformation cease to act. Such deformations are said to be *elastic*. For large deformations, the removal of the external forces does not result in the total disappearance of the deformation; a *residual deformation* remains, so that the state of the body is not that which existed before the forces were applied. Such deformations are said to be *plastic*. In what follows we shall consider only elastic deformations.

We shall also suppose that the process of deformation occurs so slowly that thermodynamic equilibrium is established in the body at every instant, in accordance with the external conditions. This assumption is almost always justified in practice. The process will then be thermodynamically reversible.

In what follows we shall take all such thermodynamic quantities as the entropy S, the internal energy $\mathscr{E}$, etc., relative to unit volume of the body,†

† The following remark should be made here. Strictly speaking, the unit volumes before and after the deformation should be distinguished, since they in general contain different amounts of matter. We shall always relate the thermodynamic quantities to unit volume of the undeformed body, i.e. to the amount of matter therein, which may occupy a different volume after the deformation. Accordingly, the total energy of the body, for example, is obtained by integrating $\mathscr{E}$ over the volume of the undeformed body.

and not relative to unit mass as in fluid mechanics, and denote them by the corresponding capital letters.

An infinitesimal change $\mathrm{d}\mathscr{E}$ in the internal energy is equal to the difference between the heat acquired by the unit volume considered and the work $\mathrm{d}R$ done by the internal stresses. The amount of heat is, for a reversible process, $T\mathrm{d}S$, where T is the temperature. Thus $\mathrm{d}\mathscr{E} = T\mathrm{d}S - \mathrm{d}R$; with $\mathrm{d}R$ given by (3.1), we obtain

$$\mathrm{d}\mathscr{E} = T\mathrm{d}S + \sigma_{ik}\,\mathrm{d}u_{ik}. \tag{3.2}$$

This is the fundamental thermodynamic relation for deformed bodies.

In hydrostatic compression, the stress tensor is $\sigma_{ik} = -p\delta_{ik}$ (2.5). Then $\sigma_{ik}\,\mathrm{d}u_{ik} = -p\delta_{ik}\,\mathrm{d}u_{ik} = -p\,\mathrm{d}u_{ii}$. We have seen, however (cf. (1.6)), that the sum u_{ii} is the relative volume change due to the deformation. If we consider unit volume, therefore, u_{ii} is simply the change in that volume, and $\mathrm{d}u_{ii}$ is the volume element $\mathrm{d}V$. The thermodynamic relation then takes its usual form

$$\mathrm{d}\mathscr{E} = T\mathrm{d}S - p\mathrm{d}V.$$

Introducing the free energy of the body, $F = \mathscr{E} - TS$, we find the form

$$\mathrm{d}F = -S\mathrm{d}T + \sigma_{ik}\,\mathrm{d}u_{ik} \tag{3.3}$$

of the relation (3.2). Finally, the thermodynamic potential Φ is defined as

$$\Phi = \mathscr{E} - TS - \sigma_{ik}u_{ik} = F - \sigma_{ik}u_{ik}. \tag{3.4}$$

This is a generalisation of the usual expression $\Phi = \mathscr{E} - TS + pV$.† Substituting (3.4) in (3.3), we find

$$\mathrm{d}\Phi = -S\mathrm{d}T - u_{ik}\,\mathrm{d}\sigma_{ik}. \tag{3.5}$$

The independent variables in (3.2) and (3.3) are respectively S, u_{ik} and T, u_{ik}. The components of the stress tensor can be obtained by differentiating $\mathscr{E}$ or F with respect to the components of the strain tensor, for constant entropy S or temperature T respectively:

$$\sigma_{ik} = (\partial\mathscr{E}/\partial u_{ik})_S = (\partial F/\partial u_{ik})_T. \tag{3.6}$$

Similarly, by differentiating Φ with respect to the components σ_{ik}, we can obtain the components u_{ik}:

$$u_{ik} = -(\partial\Phi/\partial\sigma_{ik})_T. \tag{3.7}$$

† For hydrostatic compression, the expression (3.4) becomes $\Phi = F + pu_{ii} = F + p(V - V_0)$, where $V - V_0$ is the volume change resulting from the deformation. Hence we see that the definition of Φ used here differs by a term $-pV_0$ from the usual definition $\Phi = F + pV$.

§4. Hooke's law

In order to be able to apply the general formulae of thermodynamics to any particular case, we must know the free energy F of the body as a function of the strain tensor. This expression is easily obtained by using the fact that the deformation is small and expanding the free energy in powers of u_{ik}. We shall at present consider only isotropic bodies. The corresponding results for crystals will be obtained in §10.

In considering a deformed body at some temperature (constant throughout the body), we shall take the undeformed state to be the state of the body in the absence of external forces and at the same temperature; this last condition is necessary on account of the thermal expansion (see §6). Then, for $u_{ik} = 0$, the internal stresses are zero also, i.e. $\sigma_{ik} = 0$. Since $\sigma_{ik} = \partial F/\partial u_{ik}$, it follows that there is no linear term in the expansion of F in powers of u_{ik}.

Next, since the free energy is a scalar, each term in the expansion of F must be a scalar also. Two independent scalars of the second degree can be formed from the components of the symmetrical tensor u_{ik}: they can be taken as the squared sum of the diagonal components (u_{ii}^2) and the sum of the squares of all the components (u_{ik}^2). Expanding F in powers of u_{ik}, we therefore have as far as terms of the second order

$$F = F_0+\tfrac{1}{2}\lambda u_{ii}^2+\mu u_{ik}^2. \tag{4.1}$$

This is the general expression for the free energy of a deformed isotropic body. The quantities λ and μ are called *Lamé coefficients*.

We have seen in §1 that the change in volume in the deformation is given by the sum u_{ii}. If this sum is zero, then the volume of the body is unchanged by the deformation, only its shape being altered. Such a deformation is called a *pure shear*.

The opposite case is that of a deformation which causes a change in the volume of the body but no change in its shape. Each volume element of the body retains its shape also. We have seen in §1 that the tensor of such a deformation is u_{ik} = constant $\times\delta_{ik}$. Such a deformation is called a *hydrostatic compression*.

Any deformation can be represented as the sum of a pure shear and a hydrostatic compression. To do so, we need only use the identity

$$u_{ik} = (u_{ik}-\tfrac{1}{3}\delta_{ik}u_{ll})+\tfrac{1}{3}\delta_{ik}u_{ll}. \tag{4.2}$$

The first term on the right is evidently a pure shear, since the sum of its diagonal terms is zero ($\delta_{ii} = 3$). The second term is a hydrostatic compression.

As a general expression for the free energy of a deformed isotropic body, it is convenient to replace (4.1) by another formula, using this decomposition of an arbitrary deformation into a pure shear and a hydrostatic compression. We take as the two independent scalars of the second degree the sums of the

squared components of the two terms in (4.2). Then F becomes†

$$F = \mu(u_{ik} - \tfrac{1}{3}\delta_{ik}u_{ll})^2 + \tfrac{1}{2}Ku_{ll}^2. \tag{4.3}$$

The quantities K and μ are called respectively the *bulk modulus* or *modulus of hydrostatic compression* (or simply the *modulus of compression*) and the *shear modulus* or *modulus of rigidity*. K is related to the Lamé coefficients by

$$K = \lambda + \tfrac{2}{3}\mu. \tag{4.4}$$

In a state of thermodynamic equilibrium, the free energy is a minimum. If no external forces act on the body, then F as a function of u_{ik} must have a minimum for $u_{ik} = 0$. This means that the quadratic form (4.3) must be positive. If the tensor u_{ik} is such that $u_{ll} = 0$, only the first term remains in (4.3); if, on the other hand, the tensor is of the form u_{ik} = constant $\times \delta_{ik}$, then only the second term remains. Hence it follows that a necessary (and evidently sufficient) condition for the form (4.3) to be positive is that each of the coefficients K and μ is positive. Thus we conclude that the moduli of compression and rigidity are always positive:

$$K > 0, \; \mu > 0. \tag{4.5}$$

We now use the general thermodynamic relation (3.6) to determine the stress tensor. To calculate the derivatives $\partial F/\partial u_{ik}$, we write the total differential dF (for constant temperature):

$$\mathrm{d}F = Ku_{ll}\,\mathrm{d}u_{ll} + 2\mu(u_{ik} - \tfrac{1}{3}u_{ll}\delta_{ik})\,\mathrm{d}(u_{ik} - \tfrac{1}{3}u_{ll}\delta_{ik}).$$

In the second term, multiplication of the first parenthesis by δ_{ik} gives zero, leaving $\mathrm{d}F = Ku_{ll}\,\mathrm{d}u_{ll} + 2\mu(u_{ik} - \tfrac{1}{3}u_{ll}\delta_{ik})\,\mathrm{d}u_{ik}$, or writing $\mathrm{d}u_{ll} = \delta_{ik}\,\mathrm{d}u_{ik}$,

$$\mathrm{d}F = [Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \tfrac{1}{3}u_{ll}\delta_{ik})]\,\mathrm{d}u_{ik}.$$

Hence the stress tensor is

$$\sigma_{ik} = Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \tfrac{1}{3}\delta_{ik}u_{ll}). \tag{4.6}$$

This expression determines the stress tensor in terms of the strain tensor for an isotropic body. It shows, in particular, that, if the deformation is a pure shear or a pure hydrostatic compression, the relation between σ_{ik} and u_{ik} is determined only by the modulus of rigidity or of hydrostatic compression respectively.

It is not difficult to obtain the converse formula which expresses u_{ik} in terms of σ_{ik}. To do so, we find the sum σ_{ii} of the diagonal terms. Since this sum is zero for the second term of (4.6), we have $\sigma_{ii} = 3Ku_{ii}$, or

$$u_{ii} = \sigma_{ii}/3K. \tag{4.7}$$

† The constant term F_0 is the free energy of the undeformed body, and is of no further interest. We shall therefore omit it, for brevity, taking F to be only the free energy of the deformation (the elastic free energy, as it is called).

Substituting this expression in (4.6) and so determining u_{ik}, we find

$$u_{ik} = \delta_{ik}\sigma_{ll}/9K + (\sigma_{ik} - \tfrac{1}{3}\delta_{ik}\sigma_{ll})/2\mu, \tag{4.8}$$

which gives the strain tensor in terms of the stress tensor.

Equation (4.7) shows that the relative change in volume (u_{ii}) in any deformation of an isotropic body depends only on the sum σ_{ii} of the diagonal components of the stress tensor, and the relation between u_{ii} and σ_{ii} is determined only by the modulus of hydrostatic compression. In hydrostatic compression of a body, the stress tensor is $\sigma_{ik} = -p\delta_{ik}$. Hence we have in this case, from (4.7),

$$u_{ii} = -p/K. \tag{4.9}$$

Since the deformations are small, u_{ii} and p are small quantities, and we can write the ratio u_{ii}/p of the relative volume change to the pressure in the differential form $(1/V)(\partial V/\partial p)_T$. Thus

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T.$$

The quantity $1/K$ is called the *coefficient of hydrostatic compression* (or simply the *coefficient of compression*).

We see from (4.8) that the strain tensor u_{ik} is a linear function of the stress tensor σ_{ik}. That is, the deformation is proportional to the applied forces. This law, valid for small deformations, is called *Hooke's law*.†

We may give also a useful form of the expression for the free energy of a deformed body, which is obtained immediately from the fact that F is quadratic in the strain tensor. According to EULER's theorem, $u_{ik}\partial F/\partial u_{ik} = 2F$, whence, since $\partial F/\partial u_{ik} = \sigma_{ik}$, we have

$$F = \tfrac{1}{2}\sigma_{ik}u_{ik}. \tag{4.10}$$

If we substitute in this formula the u_{ik} as linear combinations of the components σ_{ik}, the elastic energy will be represented as a quadratic function of the σ_{ik}. Again applying EULER's theorem, we obtain $\sigma_{ik}\partial F/\partial\sigma_{ik} = 2F$, and a comparison with (4.10) shows that

$$u_{ik} = \partial F/\partial\sigma_{ik}. \tag{4.11}$$

It should be emphasised, however, that, whereas the formula $\sigma_{ik} = \partial F/\partial u_{ik}$ is a general relation of thermodynamics, the inverse formula (4.11) is applicable only if HOOKE's law is valid.

† HOOKE's law is actually applicable to almost all elastic deformation. The reason is that deformations usually cease to be elastic when they are still so small that HOOKE's law is a good approximation. Substances such as rubber form an exception.

§5. Homogeneous deformations

Let us consider some simple cases of what are called *homogeneous deformations*, i.e. those in which the strain tensor is constant throughout the volume of the body. For example, the hydrostatic compression already considered is a homogeneous deformation.

We first consider a *simple extension* (or compression) of a rod. Let the rod be along the z-axis, and let forces be applied to its ends which stretch it in both directions. These forces act uniformly over the end surfaces of the rod; let the force on unit area be p.

Since the deformation is homogeneous, i.e. u_{ik} is constant through the body, the stress tensor σ_{ik} is also constant, and so it can be determined at once from the boundary conditions (2.8). There is no external force on the sides of the rod, and therefore $\sigma_{ik}n_k = 0$. Since the unit vector **n** on the side of the rod is perpendicular to the z-axis, i.e. $n_z = 0$, it follows that all the components σ_{ik} except σ_{zz} are zero. On the end surface we have $\sigma_{zi}n_i = p$, or $\sigma_{zz} = p$.

From the general expression (4.8) which relates the components of the strain and stress tensors, we see that all the components u_{ik} with $i \neq k$ are zero. For the remaining components we find

$$u_{xx} = u_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)p, \quad u_{zz} = \frac{1}{3}\left(\frac{1}{3K}+\frac{1}{\mu}\right)p. \tag{5.1}$$

The component u_{zz} gives the relative lengthening of the rod. The coefficient of p is called the *coefficient of extension*, and its reciprocal is the *modulus of extension* or *Young's modulus*, E:

$$u_{zz} = p/E, \tag{5.2}$$

where

$$E = 9K\mu/(3K+\mu). \tag{5.3}$$

The components u_{xx} and u_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called *Poisson's ratio*, σ:†

$$u_{xx} = -\sigma u_{zz}, \tag{5.4}$$

where

$$\sigma = \tfrac{1}{2}(3K-2\mu)/(3K+\mu). \tag{5.5}$$

---

† The use of σ to denote POISSON's ratio and σ_{ik} to denote the components of the stress tensor cannot lead to ambiguity, since the latter always have suffixes.

Since K and μ are always positive, Poisson's ratio can vary between -1 (for $K = 0$) and $\frac{1}{2}$ (for $\mu = 0$). Thus †

$$-1 \leqslant \sigma \leqslant \tfrac{1}{2}. \tag{5.6}$$

Finally, the relative increase in the volume of the rod is

$$u_{ii} = p/3K. \tag{5.7}$$

The free energy of a stretched rod can be obtained immediately from formula (4.10). Since only the component σ_{zz} is not zero, we have $F = \frac{1}{2}\sigma_{zz}u_{zz}$, whence

$$F = p^2/2E. \tag{5.8}$$

In what follows we shall, as is customary, use E and σ instead of K and μ. Inverting formulae (5.3) and (5.5), we have‡

$$\mu = E/2(1+\sigma), \quad K = E/3(1-2\sigma). \tag{5.9}$$

We shall write out here the general formulae of §4, with the coefficients expressed in terms of E and σ. The free energy is

$$F = \frac{E}{2(1+\sigma)}\left(u_{ik}{}^2 + \frac{\sigma}{1-2\sigma}u_{ll}{}^2\right). \tag{5.10}$$

The stress tensor is given in terms of the strain tensor by

$$\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ik}\right). \tag{5.11}$$

Conversely,

$$u_{ik} = [(1+\sigma)\sigma_{ik} - \sigma\sigma_{ll}\delta_{ik}]/E. \tag{5.12}$$

Since formulae (5.11) and (5.12) are in frequent use, we shall give them also in component form:

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\sigma)(1-2\sigma)}[(1-\sigma)u_{xx} + \sigma(u_{yy}+u_{zz})],\\
\sigma_{yy} &= \frac{E}{(1+\sigma)(1-2\sigma)}[(1-\sigma)u_{yy} + \sigma(u_{xx}+u_{zz})],\\
\sigma_{zz} &= \frac{E}{(1+\sigma)(1-2\sigma)}[(1-\sigma)u_{zz} + \sigma(u_{xx}+u_{yy})],\\
\sigma_{xy} &= \frac{E}{1+\sigma}u_{xy}, \quad \sigma_{xz} = \frac{E}{1+\sigma}u_{xz}, \quad \sigma_{yz} = \frac{E}{1+\sigma}u_{yz},
\end{aligned}\right\} \tag{5.13}$$

† In practice, POISSON's ratio varies only between 0 and $\frac{1}{2}$. There are no substances known for which $\sigma < 0$, i.e. which would expand transversely when stretched longitudinally. It may be mentioned that the inequality $\sigma > 0$ corresponds to $\lambda > 0$, where λ is the Lamé coefficient appearing in (4.1); in other words, both terms in (4.1), as well as in (4.3), are always positive in practice, although this is not thermodynamically necessary. Values of σ close to $\frac{1}{2}$ (e.g. for rubber) correspond to a modulus of rigidity which is small compared with the modulus of compression.

‡ The second Lamé coefficient is $\lambda = E\sigma/(1-2\sigma)(1+\sigma)$.

and conversely

$$\left.\begin{aligned}u_{xx} &= \frac{1}{E}[\sigma_{xx}-\sigma(\sigma_{yy}+\sigma_{zz})],\\ u_{yy} &= \frac{1}{E}[\sigma_{yy}-\sigma(\sigma_{xx}+\sigma_{zz})],\\ u_{zz} &= \frac{1}{E}[\sigma_{zz}-\sigma(\sigma_{xx}+\sigma_{yy})],\\ u_{xy} = \frac{1+\sigma}{E}\sigma_{xy},\quad u_{xz} &= \frac{1+\sigma}{E}\sigma_{xz},\quad u_{yz} = \frac{1+\sigma}{E}\sigma_{yz}.\end{aligned}\right\} \tag{5.14}$$

Let us now consider the compression of a rod whose sides are fixed in such a way that they cannot move. The external forces which cause the compression of the rod are applied to its ends and act along its length, which we again take to be along the z-axis. Such a deformation is called a *unilateral compression*. Since the rod is deformed only in the z-direction, only the component u_{zz} of u_{ik} is not zero. Then we have from (5.11)

$$\sigma_{xx} = \sigma_{yy} = \frac{E}{(1+\sigma)(1-2\sigma)}u_{zz}, \quad \sigma_{zz} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}u_{zz}.$$

Again denoting the compressing force by p ($\sigma_{zz} = p$, which is negative for a compression), we have

$$u_{zz} = p(1+\sigma)(1-2\sigma)/E(1-\sigma). \tag{5.15}$$

The coefficient of p is called the *coefficient of unilateral compression*. For the transverse stresses we have

$$\sigma_{xx} = \sigma_{yy} = p\sigma/(1-\sigma). \tag{5.16}$$

Finally, the free energy of the rod is

$$F = p^2(1+\sigma)(1-2\sigma)/2E(1-\sigma). \tag{5.17}$$

§6. Deformations with change of temperature

Let us now consider deformations which are accompanied by a change in the temperature of the body; this can occur either as a result of the deformation process itself, or from external causes.

We shall regard as the undeformed state the state of the body in the absence of external forces at some given temperature T_0. If the body is at a temperature T different from T_0, then, even if there are no external forces, it will in general be deformed, on account of thermal expansion. In the expansion of the free energy $F(T)$, there will therefore be terms linear, as well as quadratic, in the strain tensor. From the components of the tensor u_{ik}, of rank two,

we can form only one linear scalar quantity, the sum u_{ii} of its diagonal components. We shall also assume that the temperature change $T-T_0$ which accompanies the deformation is small. We can then suppose that the coefficient of u_{ii} in the expansion of F (which must vanish for $T = T_0$) is simply proportional to the difference $T-T_0$. Thus we find the free energy to be (instead of (4.3))

$$F(T) = F_0(T)-K\alpha(T-T_0)u_{ll}+\mu(u_{ik}-\tfrac{1}{3}\delta_{ik}u_{ll})^2+\tfrac{1}{2}Ku_{ll}^2, \tag{6.1}$$

where the coefficient of $T-T_0$ has been written as $-K\alpha$. The quantities μ, K and α can here be supposed constant; an allowance for their temperature dependence would lead to terms of higher order.

Differentiating F with respect to u_{ik}, we obtain the stress tensor:

$$\sigma_{ik} = -K\alpha(T-T_0)\delta_{ik}+Ku_{ll}\delta_{ik}+2\mu(u_{ik}-\tfrac{1}{3}\delta_{ik}u_{ll}). \tag{6.2}$$

The first term gives the additional stresses caused by the change in temperature. In free thermal expansion of the body (external forces being absent), there can be no internal stresses. Equating σ_{ik} to zero, we find that u_{ik} is of the form constant $\times\ \delta_{ik}$, and

$$u_{ll} = \alpha(T-T_0). \tag{6.3}$$

But u_{ll} is the relative change in volume caused by the deformation. Thus α is just the *thermal expansion coefficient* of the body.

Among the various (thermodynamic) types of deformation, isothermal and adiabatic deformations are of importance. In isothermal deformations, the temperature of the body does not change. Accordingly, we must put $T = T_0$ in (6.1), returning to the usual formulae; the coefficients K and μ may therefore be called *isothermal moduli.*

A deformation is adiabatic if there is no exchange of heat between the various parts of the body (or, of course, between the body and the surrounding medium). The entropy S remains constant. It is the derivative $-\partial F/\partial T$ of the free energy with respect to temperature. Differentiating the expression (6.1), we have as far as terms of the first order in u_{ik}

$$S(T) = S_0(T)+K\alpha u_{ll}. \tag{6.4}$$

Putting S constant, we can determine the change of temperature $T-T_0$ due to the deformation, which is therefore proportional to u_{ll}. Substituting this expression for $T-T_0$ in (6.2), we obtain for σ_{ik} an expression of the usual kind,

$$\sigma_{ik} = K_{\mathrm{ad}}u_{ll}\delta_{ik}+2\mu(u_{ik}-\tfrac{1}{3}\delta_{ik}u_{ll}), \tag{6.5}$$

with the same modulus of rigidity μ but a different modulus of compression K_{ad}. The relation between the adiabatic modulus K_{ad} and the ordinary isothermal modulus K can also be found directly from the thermodynamic formula

$$\left(\frac{\partial V}{\partial p}\right)_S = \left(\frac{\partial V}{\partial p}\right)_T + \frac{T(\partial V/\partial T)_p{}^2}{C_p},$$

where C_p is the specific heat per unit volume at constant pressure. If V is taken to be the volume occupied by matter which before the deformation occupied unit volume, the derivatives $\partial V/\partial T$ and $\partial V/\partial p$ give the relative volume changes in heating and compression respectively. That is,

$$(\partial V/\partial T)_p = \alpha, \quad (\partial V/\partial p)_S = -1/K_{\text{ad}}, \quad (\partial V/\partial p)_T = -1/K.$$

Thus we find the relation between the adiabatic and isothermal moduli to be

$$1/K_{\text{ad}} = 1/K - T\alpha^2/C_p, \quad \mu_{\text{ad}} = \mu. \tag{6.6}$$

For the adiabatic YOUNG's modulus and POISSON's ratio we easily obtain

$$E_{\text{ad}} = \frac{E}{1-ET\alpha^2/9C_p}, \qquad \sigma_{\text{ad}} = \frac{\sigma+ET\alpha^2/9C_p}{1-ET\alpha^2/9C_p}. \tag{6.7}$$

In practice, $ET\alpha^2/C_p$ is usually small, and it is therefore sufficiently accurate to put

$$E_{\text{ad}} = E+E^2T\alpha^2/9C_p, \qquad \sigma_{\text{ad}} = \sigma+(1+\sigma)ET\alpha^2/9C_p. \tag{6.8}$$

In isothermal deformation, the stress tensor is given in terms of the derivatives of the free energy:

$$\sigma_{ik} = (\partial F/\partial u_{ik})_T.$$

For constant entropy, on the other hand, we have (see (3.6))

$$\sigma_{ik} = (\partial \mathscr{E}/\partial u_{ik})_S,$$

where $\mathscr{E}$ is the internal energy. Accordingly, the expression analogous to (4.3) determines, for adiabatic deformations, not the free energy but the internal energy per unit volume:

$$\mathscr{E} = \tfrac{1}{2}K_{\text{ad}}u_{ll}^2+\mu(u_{ik}-\tfrac{1}{3}u_{ll}\delta_{ik})^2. \tag{6.9}$$

§7. The equations of equilibrium for isotropic bodies

Let us now derive the equations of equilibrium for isotropic solid bodies. To do so, we substitute in the general equations (2.7)

$$\partial\sigma_{ik}/\partial x_k+\rho g_i = 0$$

the expression (5.11) for the stress tensor. We have

$$\frac{\partial\sigma_{ik}}{\partial x_k} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\frac{\partial u_{ll}}{\partial x_i}+\frac{E}{1+\sigma}\frac{\partial u_{ik}}{\partial x_k}.$$

Substituting

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right),$$

we obtain the equations of equilibrium in the form

$$\frac{E}{2(1+\sigma)}\frac{\partial^2 u_i}{\partial x_k^2}+\frac{E}{2(1+\sigma)(1-2\sigma)}\frac{\partial^2 u_l}{\partial x_i \partial x_l}+\rho g_i = 0. \tag{7.1}$$

These equations can be conveniently rewritten in vector notation. The quantities $\partial^2 u_i/\partial x_k^2$ are components of the vector $\triangle \mathbf{u}$, and $\partial u_l/\partial x_l \equiv \operatorname{div} \mathbf{u}$. Thus the equations of equilibrium become

$$\triangle \mathbf{u}+\frac{1}{1-2\sigma}\,\mathbf{grad}\operatorname{div}\mathbf{u} = -\rho\mathbf{g}\frac{2(1+\sigma)}{E}. \tag{7.2}$$

It is sometimes useful to transform this equation by using the vector identity $\mathbf{grad}\operatorname{div}\mathbf{u} = \triangle\mathbf{u}+\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}$. Then (7.2) becomes

$$\mathbf{grad}\operatorname{div}\mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = -\rho\mathbf{g}\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \tag{7.3}$$

We have written the equations of equilibrium for a uniform gravitational field, since this is the body force most usually encountered in the theory of elasticity. If there are other body forces, the vector $\rho\mathbf{g}$ on the right-hand side of the equation must be replaced accordingly.

A very important case is that where the deformation of the body is caused, not by body forces, but by forces applied to its surface. The equation of equilibrium then becomes

$$(1-2\sigma)\triangle\mathbf{u}+\mathbf{grad}\operatorname{div}\mathbf{u} = 0 \tag{7.4}$$

or

$$2(1-\sigma)\,\mathbf{grad}\operatorname{div}\mathbf{u}-(1-2\sigma)\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0. \tag{7.5}$$

The external forces appear in the solution only through the boundary conditions.

Taking the divergence of equation (7.4) and using the identity

$$\operatorname{div}\mathbf{grad} \equiv \triangle,$$

we find

$$\triangle \operatorname{div}\mathbf{u} = 0, \tag{7.6}$$

i.e. $\operatorname{div}\mathbf{u}$ (which determines the volume change due to the deformation) is a harmonic function. Taking the Laplacian of equation (7.4), we then obtain

$$\triangle\triangle\mathbf{u} = 0, \tag{7.7}$$

i.e. in equilibrium the displacement vector satisfies the *biharmonic equation*. These results remain valid in a uniform gravitational field (since the right-hand side of equation (7.2) gives zero on differentiation), but not in the general case of external forces which vary through the body.

The fact that the displacement vector satisfies the biharmonic equation does not, of course, mean that the general integral of the equations of equilibrium (in the absence of body forces) is an arbitrary biharmonic vector; it must be remembered that the function $\mathbf{u}(x, y, z)$ also satisfies the lower order differential equation (7.4). It is possible, however, to express the general integral of the equations of equilibrium in terms of the derivatives of an arbitrary biharmonic vector (see Problem 10).

If the body is non-uniformly heated, an additional term appears in the equation of equilibrium. The stress tensor must include the term

$$-K\alpha(T-T_0)\delta_{ik}$$

(see (6.2)), and $\partial\sigma_{ik}/\partial x_k$ accordingly contains a term

$$-K\alpha\partial T/\partial x_i = -[E\alpha/3(1-2\sigma)]\partial T/\partial x_i.$$

The equation of equilibrium thus takes the form

$$\frac{3(1-\sigma)}{1+\sigma}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u} - \frac{3(1-2\sigma)}{2(1+\sigma)}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \alpha\,\mathbf{grad}\,T. \tag{7.8}$$

Let us consider the particular case of a *plane deformation*, in which one component of the displacement vector (u_z) is zero throughout the body, while the components u_x, u_y depend only on x and y. The components u_{zz}, u_{xz}, u_{yz} of the strain tensor then vanish identically, and therefore so do the components σ_{xz}, σ_{yz} of the stress tensor (but not the longitudinal stress σ_{zz}, the existence of which is implied by the constancy of the length of the body in the z-direction).†

Since all quantities are independent of the co-ordinate z, the equations of equilibrium (in the absence of external body-forces) $\partial\sigma_{ik}/\partial x_k = 0$ reduce in this case to two equations:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y} = 0. \tag{7.9}$$

The most general functions σ_{xx}, σ_{xy}, σ_{yy} satisfying these equations are of the form

$$\sigma_{xx} = \partial^2\chi/\partial y^2, \qquad \sigma_{xy} = -\partial^2\chi/\partial x\partial y, \qquad \sigma_{yy} = \partial^2\chi/\partial x^2, \tag{7.10}$$

where χ is an arbitrary function of x and y. It is easy to obtain an equation which must be satisfied by this function. Such an equation must exist, since the three quantities σ_{xx}, σ_{xy}, σ_{yy} can be expressed in terms of the two quantities u_x, u_y, and are therefore not independent. Using formulae (5.13), we find, for a plane deformation,

$$\sigma_{xx}+\sigma_{yy} = E(u_{xx}+u_{yy})/(1+\sigma)(1-2\sigma).$$

---

† The use of the theory of functions of a complex variable provides very powerful methods of solving plane problems in the theory of elasticity. See N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd English ed., P. Noordhoff, Groningen 1963.

But

$$\sigma_{xx}+\sigma_{yy} = \triangle\chi, \qquad u_{xx}+u_{yy} = \frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y} \equiv \operatorname{div}\mathbf{u},$$

and, since by (7.6) div **u** is harmonic, we conclude that the function χ satisfies the equation

$$\triangle\triangle\chi = 0, \tag{7.11}$$

i.e. it is biharmonic. This function is called the *stress function*. When the plane problem has been solved and the function χ is known, the longitudinal stress σ_{zz} is determined at once from the formula

$$\sigma_{zz} = \sigma E(u_{xx}+u_{yy})/(1+\sigma)(1-2\sigma) = \sigma(\sigma_{xx}+\sigma_{yy}),$$

or

$$\sigma_{zz} = \sigma\triangle\chi. \tag{7.12}$$

PROBLEMS

PROBLEM 1. Determine the deformation of a long rod (of length l) standing vertically in a gravitational field.

SOLUTION. We take the z-axis along the axis of the rod, and the xy-plane in the plane of its lower end. The equations of equilibrium are $\partial\sigma_{xi}/\partial x_i = \partial\sigma_{yi}/\partial x_i = 0$, $\partial\sigma_{zi}/\partial x_i = \rho g$. On the sides of the rod all the components σ_{ik} except σ_{zz} must vanish, and on the upper end ($z = l$) $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. The solution of the equations of equilibrium satisfying these conditions is $\sigma_{zz} = -\rho g(l-z)$, with all other σ_{ik} zero. From σ_{ik} we find u_{ik} to be $u_{xx} = u_{yy} = \sigma\rho g(l-z)/E$, $u_{zz} = -\rho g(l-z)/E$, $u_{xy} = u_{xz} = u_{yz} = 0$, and hence by integration we have the components of the displacement vector, $u_x = \sigma\rho g(l-z)x/E$, $u_y = \sigma\rho g(l-z)y/E$, $u_z = -(\rho g/2E)\{l^2-(l-z)^2-\sigma(x^2+y^2)\}$. The expression for u_z satisfies the boundary condition $u_z = 0$ only at one point on the lower end of the rod. Hence the solution obtained is not valid near the lower end.

PROBLEM 2. Determine the deformation of a hollow sphere (of external and internal radii R_2 and R_1) with a pressure p_1 inside and p_2 outside.

SOLUTION. We use spherical co-ordinates, with the origin at the centre of the sphere. The displacement vector **u** is everywhere radial, and is a function of r alone. Hence **curl u**$=0$, and equation (7.5) becomes **grad** div **u** $= 0$. Hence

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2}\frac{\mathrm{d}(r^2u)}{\mathrm{d}r} = \text{constant} \equiv 3a,$$

or $u = ar+b/r^2$. The components of the strain tensor are (see formulae (1.7)) $u_{rr} = a-2b/r^3$, $u_{\theta\theta} = u_{\phi\phi} = a+b/r^3$. The radial stress is

$$\sigma_{rr} = \frac{E}{(1+\sigma)(1-2\sigma)}\{(1-\sigma)u_{rr}+2\sigma u_{\theta\theta}\} = \frac{E}{1-2\sigma}a - \frac{2E}{1+\sigma}\frac{b}{r^3}.$$

The constants a and b are determined from the boundary conditions: $\sigma_{rr} = -p_1$ at $r = R_1$, and $\sigma_{rr} = -p_2$ at $r = R_2$. Hence we find

$$a = \frac{p_1R_1^3-p_2R_2^3}{R_2^3-R_1^3}\frac{1-2\sigma}{E}, \qquad b = \frac{R_1^3R_2^3(p_1-p_2)}{R_2^3-R_1^3}\frac{1+\sigma}{2E}.$$

For example, the stress distribution in a spherical shell with a pressure $p_1 = p$ inside and $p_2 = 0$ outside is given by

$$\sigma_{rr} = \frac{pR_1^3}{R_2^3-R_1^3}\left(1-\frac{R_2^3}{r^3}\right), \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{pR_1^3}{R_2^3-R_1^3}\left(1+\frac{R_2^3}{2r^3}\right).$$

For a thin spherical shell of thickness $h = R_2-R_1 \ll R$ we have approximately

$$u = pR^2(1-\sigma)/2Eh, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \tfrac{1}{2}pR/h, \qquad \bar{\sigma}_{rr} = -\tfrac{1}{2}p,$$

where $\bar{\sigma}_{rr}$ is the mean value of the radial stress over the thickness of the shell.

The stress distribution in an infinite elastic medium with a spherical cavity (of radius R) subjected to hydrostatic compression is obtained by putting $R_1 = R$, $R_2 = \infty$, $p_1 = 0$, $p_2 = p$:

$$\sigma_{rr} = -p\left(1-\frac{R^3}{r^3}\right), \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p\left(1+\frac{R^3}{2r^3}\right).$$

At the surface of the cavity the tangential stresses $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -3p/2$, i.e. they exceed the pressure at infinity.

PROBLEM 3. Determine the deformation of a solid sphere (of radius R) in its own gravitational field.

SOLUTION. The force of gravity on unit mass in a spherical body is $-\mathbf{g}\mathbf{r}/R$. Substituting this expression in place of $\mathbf{g}$ in equation (7.3), we obtain the following equation for the radial displacement:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\frac{\mathrm{d}(r^2u)}{\mathrm{d}r}\right) = \rho g\frac{r}{R}.$$

The solution finite for $r = 0$ which satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = -\frac{g\rho R(1-2\sigma)(1+\sigma)}{10E(1-\sigma)}r\left(\frac{3-\sigma}{1+\sigma}-\frac{r^2}{R^2}\right).$$

It should be noticed that the substance is compressed ($u_{rr} < 0$) inside a spherical surface of radius $R\sqrt{\{(3-\sigma)/3(1+\sigma)\}}$ and stretched outside it ($u_{rr} > 0$). The pressure at the centre of the sphere is $(3-\sigma)g\rho R/10(1-\sigma)$.

PROBLEM 4. Determine the deformation of a cylindrical pipe (of external and internal radii R_2 and R_1), with a pressure p inside and no pressure outside.†

SOLUTION. We use cylindrical co-ordinates, with the z-axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $u_r = u(r)$. Similarly to Problem 2, we have

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\mathrm{d}(ru)}{\mathrm{d}r} = \text{constant} \equiv 2a.$$

Hence $u = ar+b/r$. The non-zero components of the strain tensor are (see formulae (1.8)) $u_{rr} = \mathrm{d}u/\mathrm{d}r = a-b/r^2$, $u_{\phi\phi} = u/r = a+b/r^2$. From the conditions $\sigma_{rr} = 0$ at $r = R_2$, and $\sigma_{rr} = -p$ at $r = R_1$, we find

$$a = \frac{pR_1^2}{R_2^2-R_1^2}\cdot\frac{(1+\sigma)(1-2\sigma)}{E}, \qquad b = \frac{pR_1^2R_2^2}{R_2^2-R_1^2}\cdot\frac{1+\sigma}{E}.$$

† In Problems 4, 5 and 7 it is assumed that the length of the cylinder is maintained constant, so that there is no longitudinal deformation.

The stress distribution is given by the formulae

$$\sigma_{rr} = \frac{pR_1^2}{R_2^2-R_1^2}\left(1-\frac{R_2^2}{r^2}\right), \qquad \sigma_{\phi\phi} = \frac{pR_1^2}{R_2^2-R_1^2}\left(1+\frac{R_2^2}{r^2}\right),$$

$$\sigma_{zz} = 2p\sigma R_1^2/(R_2^2-R_1^2).$$

PROBLEM 5. Determine the deformation of a cylinder rotating uniformly about its axis.

SOLUTION. Replacing the gravitational force in (7.3) by the centrifugal force $\rho\Omega^2 r$ (where Ω is the angular velocity), we have in cylindrical co-ordinates the following equation for the displacement $u_r = u(r)$:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}(ru)}{\mathrm{d}r}\right) = -\rho\Omega^2 r.$$

The solution which is finite for $r = 0$ and satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = \frac{\rho\Omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}r[(3-2\sigma)R^2-r^2].$$

PROBLEM 6. Determine the deformation of a non-uniformly heated sphere with a spherically symmetrical temperature distribution.

SOLUTION. In spherical co-ordinates, equation (7.8) for a purely radial deformation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\frac{\mathrm{d}(r^2u)}{\mathrm{d}r}\right) = \alpha\frac{1+\sigma}{3(1-\sigma)}\frac{\mathrm{d}T}{\mathrm{d}r}.$$

The solution which is finite for $r = 0$ and satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = \alpha\frac{1+\sigma}{3(1-\sigma)}\left\{\frac{1}{r^2}\int_0^r T(r)r^2\,\mathrm{d}r + \frac{2(1-2\sigma)}{1+\sigma}\frac{r}{R^3}\int_0^R T(r)r^2\,\mathrm{d}r\right\}.$$

The temperature $T(r)$ is measured from the value for which the sphere, if uniformly heated, is regarded as undeformed. In the above formula the temperature in question is taken as that of the outer surface of the sphere, so that $T(R) = 0$.

PROBLEM 7. The same as Problem 6, but for a non-uniformly heated cylinder with an axially symmetrical temperature distribution.

SOLUTION. We similarly have in cylindrical co-ordinates

$$u = \alpha\frac{1+\sigma}{3(1-\sigma)}\left\{\frac{1}{r}\int_0^r T(r)r\,\mathrm{d}r + (1-2\sigma)\frac{r}{R^2}\int_0^R T(r)r\,\mathrm{d}r\right\}.$$

PROBLEM 8. Determine the deformation of an infinite elastic medium with a given temperature distribution $T(x, y, z)$ which is such that the temperature tends to a constant value T_0 at infinity, there being no deformation there.

SOLUTION. Equation (7.8) has an obvious solution for which $\mathbf{curl\,u} = 0$ and

$$\mathrm{div}\,\mathbf{u} = \alpha(1+\sigma)[T(x,y,z)-T_0]/3(1-\sigma).$$

The vector $\mathbf{u}$, whose divergence is a given function defined in all space and vanishing at infinity, and whose curl is zero identically, can be written, as we know from vector analysis, in the form

$$\mathbf{u}(x, y, z) = -\frac{1}{4\pi}\mathbf{grad}\int\frac{\operatorname{div}'\mathbf{u}(x', y', z')}{r}\mathrm{d}V',$$

where

$$r = \sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}.$$

We therefore obtain the general solution of the problem in the form

$$\mathbf{u} = -\frac{\alpha(1+\sigma)}{12\pi(1-\sigma)}\mathbf{grad}\int\frac{T'-T_0}{r}\mathrm{d}V', \tag{1}$$

where $T' \equiv T(x', y', z')$.

If a finite quantity of heat q is evolved in a very small volume at the origin, the temperature distribution can be written $T-T_0 = (q/C)\delta(x)\delta(y)\delta(z)$, where C is the specific heat of the medium. The integral in (1) is then q/Cr, and the deformation is given by

$$\mathbf{u} = \frac{\alpha(1+\sigma)q}{12\pi(1-\sigma)C}\cdot\frac{\mathbf{r}}{r^3}.$$

PROBLEM 9. Derive the equations of equilibrium for an isotropic body (in the absence of body forces) in terms of the components of the stress tensor.

SOLUTION. The required system of equations contains the three equations

$$\partial\sigma_{ik}/\partial x_k = 0 \tag{1}$$

and also the equations resulting from the fact that the six different components of u_{ik} are not independent quantities. To derive these equations, we first write down the system of differential relations satisfied by the components of the tensor u_{ik}. It is easy to see that the quantities

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)$$

satisfy identically the relations

$$\frac{\partial^2 u_{ik}}{\partial x_l\partial x_m}+\frac{\partial^2 u_{lm}}{\partial x_i\partial x_k} = \frac{\partial^2 u_{il}}{\partial x_k\partial x_m}+\frac{\partial^2 u_{km}}{\partial x_i\partial x_l}.$$

Here there are only six essentially different relations, namely those corresponding to the following values of i, k, l, m: 1122, 1133, 2233, 1123, 2213, 3312. All these are retained if the above tensor equation is contracted with respect to l and m:

$$\triangle u_{ik}+\frac{\partial^2 u_{ll}}{\partial x_i\partial x_k} = \frac{\partial^2 u_{il}}{\partial x_k\partial x_l}+\frac{\partial^2 u_{kl}}{\partial x_i\partial x_l}. \tag{2}$$

Substituting here u_{ik} in terms of σ_{ik} according to (5.12) and using (1), we obtain the required equations:

$$(1+\sigma)\triangle\sigma_{ik}+\frac{\partial^2\sigma_{ll}}{\partial x_i\partial x_k} = 0. \tag{3}$$

These equations remain valid in the presence of external forces constant throughout the body.

Contracting equation (3) with respect to the suffixes i and k, we find that $\triangle \sigma_{ll} = 0$, i.e. σ_{ll} is a harmonic function. Taking the Laplacian of equation (3), we then find that $\triangle\triangle\sigma_{ik} = 0$, i.e. the components σ_{ik} are biharmonic functions. These results follow also from (7.6) and (7.7), since σ_{ik} and u_{ik} are linearly related.

PROBLEM 10. Express the general integral of the equations of equilibrium (in the absence of body forces) in terms of an arbitrary biharmonic vector (B. G. GALERKIN 1930).

SOLUTION. It is natural to seek a solution of equation (7.4) in the form

$$\mathbf{u} = \triangle \mathbf{f} + A\,\mathbf{grad}\,\mathrm{div}\,\mathbf{f}.$$

Hence $\mathrm{div}\,\mathbf{u} = (1+A)\,\mathrm{div}\,\triangle\mathbf{f}$. Substituting in (7.4), we obtain

$$(1-2\sigma)\triangle\triangle\mathbf{f} + [2(1-\sigma)A+1]\,\mathbf{grad}\,\mathrm{div}\,\triangle\mathbf{f} = 0.$$

From this we see that, if $\mathbf{f}$ is an arbitrary biharmonic vector ($\triangle\triangle\mathbf{f} = 0$), then

$$\mathbf{u} = \triangle\mathbf{f} - \frac{1}{2(1-\sigma)}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{f}.$$

PROBLEM 11. Express the stresses σ_{rr}, $\sigma_{\phi\phi}$, $\sigma_{r\phi}$ for a plane deformation (in polar co-ordinates r, ϕ) as derivatives of the stress function.

SOLUTION. Since the required expressions cannot depend on the choice of the initial line of ϕ, they do not contain ϕ explicitly. Hence we can proceed as follows: we transform the Cartesian derivatives (7.10) into derivatives with respect to r, ϕ, and use the results that $\sigma_{rr} = (\sigma_{xx})_{\phi=0}$, $\sigma_{\phi\phi} = (\sigma_{yy})_{\phi=0}$, $\sigma_{r\phi} = (\sigma_{xy})_{\phi=0}$, the angle ϕ being measured from the x-axis. Thus

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2}, \qquad \sigma_{\phi\phi} = \frac{\partial^2\chi}{\partial r^2}, \qquad \sigma_{r\phi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\phi}\right).$$

PROBLEM 12. Determine the stress distribution in an infinite elastic medium containing a spherical cavity and subjected to a homogeneous deformation at infinity.

SOLUTION. A general homogeneous deformation can be represented as a combination of a homogeneous hydrostatic extension (or compression) and a homogeneous shear. The former has been considered in Problem 2, so that we need only consider a homogeneous shear. Let $\sigma_{ik}^{(0)}$ be the homogeneous stress field which would be found in all space if the cavity were absent: in a pure shear $\sigma_{ii}^{(0)} = 0$. The corresponding displacement vector is denoted by $\mathbf{u}^{(0)}$, and we seek the required solution in the form $\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$, where the function $\mathbf{u}^{(1)}$ arising from the presence of the cavity is zero at infinity.

Any solution of the biharmonic equation can be written as a linear combination of centrally symmetrical solutions and their spatial derivatives of various orders. The functions r^2, r, 1, $1/r$ are independent centrally symmetrical solutions. Hence the most general form of a biharmonic vector $\mathbf{u}^{(1)}$, depending only on the components of the constant tensor $\sigma_{ik}^{(0)}$ as parameters and vanishing at infinity, is

$$u_i^{(1)} = A\sigma_{ik}^{(0)}\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) + B\sigma_{kl}^{(0)}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}\left(\frac{1}{r}\right) + C\sigma_{kl}^{(0)}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}r. \tag{1}$$

Substituting this expression in equation (7.4), we obtain

$$(1-2\sigma)\frac{\partial^2 u_i}{\partial x_l^2} + \frac{\partial}{\partial x_i}\frac{\partial u_l}{\partial x_l} = [2(1-2\sigma)C + (A+2C)]\sigma_{kl}^{(0)}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}\frac{1}{r} = 0,$$

whence $A = -4C(1-\sigma)$. Two further relations between the constants A, B, C are obtained from the condition at the surface of the cavity: $(\sigma_{ik}^{(0)} + \sigma_{ik}^{(1)})n_k = 0$ for $r = R$ (R being the

radius of the cavity, the origin at its centre, and **n** a unit vector parallel to **r**). A somewhat lengthy calculation, using (1), gives the following values:

$$B = CR^2/5, \qquad C = 5R^3(1+\sigma)/2E(7-5\sigma).$$

The final expression for the stress distribution is

$$\sigma_{ik} = \sigma_{ik}^{(0)}\left\{1+\frac{5(1-2\sigma)}{7-5\sigma}\left(\frac{R}{r}\right)^3+\frac{3}{7-5\sigma}\left(\frac{R}{r}\right)^5\right\}+$$

$$+\frac{15}{7-5\sigma}\left(\frac{R}{r}\right)^3\left\{\sigma-\left(\frac{R}{r}\right)^2\right\}(\sigma_{il}^{(0)}n_kn_l+\sigma_{kl}^{(0)}n_ln_i)+$$

$$+\frac{15}{2(7-5\sigma)}\left(\frac{R}{r}\right)^3\left\{-5+7\left(\frac{R}{r}\right)^2\right\}\sigma_{lm}^{(0)}n_ln_mn_in_k+$$

$$+\frac{15}{2(7-5\sigma)}\left(\frac{R}{r}\right)^3\left\{1-2\sigma-\left(\frac{R}{r}\right)^2\right\}\delta_{ik}\sigma_{lm}^{(0)}n_ln_m.$$

In order to obtain the stress distribution for arbitrary $\sigma_{ik}^{(0)}$ (not a pure shear), $\sigma_{ik}^{(0)}$ in this expression must be replaced by $\sigma_{ik}^{(0)}-\frac{1}{3}\delta_{ik}\,\sigma_{ll}^{(0)}$, and the expression

$$\tfrac{1}{3}\sigma_{ll}^{(0)}\left[\delta_{ik}+\frac{R^3}{2r^3}(\delta_{ik}-3n_in_k)\right]$$

corresponding to a deformation homogeneous at infinity (cf. Problem 2) must be added. We may give here the general formula for the stresses at the surface of the cavity:

$$\sigma_{ik} = \frac{15}{7-5\sigma}\left\{(1-\sigma)(\sigma_{ik}^{(0)}-\sigma_{il}^{(0)}n_ln_k-\sigma_{kl}^{(0)}n_ln_i)+\right.$$

$$\left.+\sigma_{lm}^{(0)}n_ln_mn_in_k-\sigma\sigma_{lm}^{(0)}n_ln_m\delta_{ik}+\frac{5\sigma-1}{10}\sigma_{ll}^{(0)}(\delta_{ik}-n_in_k)\right\}$$

Near the cavity, the stresses considerably exceed the stresses at infinity, but this extends over only a short distance (the *concentration of stresses*). For example, if the medium is subjected to a homogeneous extension (only $\sigma_{zz}^{(0)}$ different from zero), the greatest stress occurs on the equator of the cavity, where

$$\sigma_{zz} = \frac{27-15\sigma}{2(7-5\sigma)}\sigma_{zz}^{(0)}.$$

§8. Equilibrium of an elastic medium bounded by a plane

Let us consider an elastic medium occupying a half-space, i.e. bounded on one side by an infinite plane, and determine the deformation of the

medium caused by forces applied to its free surface.† The distribution of these forces need satisfy only one condition: they must vanish at infinity in such a way that there is no deformation at infinity. In such a case the equations of equilibrium can be integrated in a general form.

The equation of equilibrium (7.4) holds throughout the space occupied by the medium:

$$\mathbf{grad}\,\mathrm{div}\,\mathbf{u}+(1-2\sigma)\triangle\mathbf{u} = 0. \tag{8.1}$$

We seek a solution of this equation in the form

$$\mathbf{u} = \mathbf{f}+\mathbf{grad}\,\phi, \tag{8.2}$$

where ϕ is some scalar and the vector $\mathbf{f}$ satisfies LAPLACE's equation:

$$\triangle\mathbf{f} = 0. \tag{8.3}$$

Substituting (8.2) in (8.1), we then obtain the following equation for ϕ:

$$2(1-\sigma)\triangle\phi = -\,\mathrm{div}\,\mathbf{f}. \tag{8.4}$$

We take the free surface of the elastic medium as the xy-plane; the medium is in $z > 0$. We write the functions f_x and f_y as the z-derivatives of some functions g_x and g_y:

$$f_x = \partial g_x/\partial z, \qquad f_y = \partial g_y/\partial z. \tag{8.5}$$

Since f_x and f_y are harmonic functions, we can always choose the functions g_x and g_y so as to satisfy LAPLACE's equation:

$$\triangle g_x = 0, \qquad \triangle g_y = 0. \tag{8.6}$$

Equation (8.4) then becomes

$$2(1-\sigma)\triangle\phi = -\frac{\partial}{\partial z}\left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}+f_z\right).$$

Since g_x, g_y and f_z are harmonic functions, we easily see that a function ϕ which satisfies this equation can be written as

$$\phi = -\frac{z}{4(1-\sigma)}\left(f_z+\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+\psi, \tag{8.7}$$

where ψ is again a harmonic function:

$$\triangle\psi = 0. \tag{8.8}$$

† The most direct and regular method of solving this problem is to use FOURIER's method on equation (8.1). In that case, however, some fairly complicated integrals have to be calculated. The method given below is based on a number of artificial devices, but the calculations are simpler.

Thus the problem of determining the displacement **u** reduces to that of finding the functions g_x, g_y, f_z, ψ, all of which satisfy LAPLACE's equation.

We shall now write out the boundary conditions which must be satisfied at the free surface of the medium (the plane $z = 0$). Since the unit outward normal vector **n** is in the negative z-direction, it follows from the general formula (2.8) that $\sigma_{iz} = -P_i$. Using for σ_{ik} the general expression (5.11) and expressing the components of the vector **u** in terms of the auxiliary quantities g_x, g_y, f_z and ψ, we obtain after a simple calculation the boundary conditions

$$\left[\frac{\partial^2 g_x}{\partial z^2}\right]_{z=0} + \left[\frac{\partial}{\partial x}\left\{\frac{1-2\sigma}{2(1-\sigma)}f_z - \frac{1}{2(1-\sigma)}\left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -2(1+\sigma)P_x/E, \qquad (8.9)$$

$$\left[\frac{\partial^2 g_y}{\partial z^2}\right]_{z=0} + \left[\frac{\partial}{\partial y}\left\{\frac{1-2\sigma}{2(1-\sigma)}f_z - \frac{1}{2(1-\sigma)}\left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -2(1+\sigma)P_y/E,$$

$$\left[\frac{\partial}{\partial z}\left\{f_z - \left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -2(1+\sigma)P_z/E. \qquad (8.10)$$

The components P_x, P_y, P_z of the external forces applied to the surface are given functions of the co-ordinates x and y, and vanish at infinity.

The formulae by which the auxiliary quantities g_x, g_y, f_z and ψ were defined do not determine them uniquely. We can therefore impose an arbitrary additional condition on these quantities, and it is convenient to make the quantity in the braces in equations (8.9) vanish:†

$$(1-2\sigma)f_z - \left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+4(1-\sigma)\frac{\partial \psi}{\partial z} = 0. \qquad (8.11)$$

Then the conditions (8.9) become simply

$$\left[\frac{\partial^2 g_x}{\partial z^2}\right]_{z=0} = -\frac{2(1+\sigma)}{E}P_x, \quad \left[\frac{\partial^2 g_y}{\partial z^2}\right]_{z=0} = -\frac{2(1+\sigma)}{E}P_y. \qquad (8.12)$$

Equations (8.10)–(8.12) suffice to determine completely the harmonic functions g_x, g_y, f_z and ψ.

For simplicity, we shall consider the case where the free surface of an elastic half-space is subjected to a concentrated force **F**, i.e. one which is applied to an area so small that it can be regarded as a point. The effect of this force is the same as that of surface forces given by $\mathbf{P} = \mathbf{F}\delta(x)\delta(y)$, the origin being at the point of application of the force. If we know the solution for a concentrated force, we can immediately find the solution for any force distribution $\mathbf{P}(x, y)$. For, if

$$u_i = G_{ik}(x, y, z)F_k \qquad (8.13)$$

† We shall not prove here that this condition can in fact be imposed; this follows from the absence of contradiction in the result.

is the displacement due to the action of a concentrated force $\mathbf{F}$ applied at the origin, then the displacement caused by forces $\mathbf{P}(x, y)$ is given by the integral†

$$u_i = \int\int G_{ik}(x-x', y-y', z)P_k(x', y')\, dx'\, dy'. \tag{8.14}$$

We know from potential theory that a harmonic function f which is zero at infinity and has a given normal derivative $\partial f/\partial z$ on the plane $z = 0$ is given by the formula

$$f(x, y, z) = -\frac{1}{2\pi}\int\int\left[\frac{\partial f(x', y', z)}{\partial z}\right]_{z=0}\frac{dx'\, dy'}{r},$$

where

$$r = \sqrt{\{(x-x')^2+(y-y')^2+z^2\}}.$$

Since the quantities $\partial g_x/\partial z$, $\partial g_y/\partial z$ and that in the braces in equation (8.10) satisfy LAPLACE's equation, while equations (8.10) and (8.12) determine the values of their normal derivatives on the plane $z = 0$, we have

$$\begin{aligned} f_z - \left(\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}\right)+2\frac{\partial\psi}{\partial z} &= \frac{1+\sigma}{\pi E}\int\int\frac{P_z(x', y')}{r}\, dx'\, dy' \\ &= \frac{1+\sigma}{\pi E}\cdot\frac{F_z}{r}, \end{aligned} \tag{8.15}$$

$$\frac{\partial g_x}{\partial z} = \frac{1+\sigma}{\pi E}\cdot\frac{F_x}{r}, \qquad \frac{\partial g_y}{\partial z} = \frac{1+\sigma}{\pi E}\cdot\frac{F_y}{r}, \tag{8.16}$$

where now $r = \sqrt{(x^2+y^2+z^2)}$.

The expressions for the components of the required vector $\mathbf{u}$ involve the derivatives of g_x, g_y with respect to x, y, z, but not g_x, g_y themselves. To calculate $\partial g_x/\partial x$, $\partial g_y/\partial y$, we differentiate equations (8.16) with respect to x and y respectively:

$$\frac{\partial^2 g_x}{\partial x\partial z} = -\frac{1+\sigma}{\pi E}\cdot\frac{F_x x}{r^3}, \qquad \frac{\partial^2 g_y}{\partial y\partial z} = -\frac{1+\sigma}{\pi E}\cdot\frac{F_y y}{r^3}.$$

Now, integrating over z from ∞ to z, we obtain

$$\begin{aligned} \frac{\partial g_x}{\partial x} &= \frac{1+\sigma}{\pi E}\cdot\frac{F_x x}{r(r+z)}, \\ \frac{\partial g_y}{\partial y} &= \frac{1+\sigma}{\pi E}\cdot\frac{F_y y}{r(r+z)}. \end{aligned} \tag{8.17}$$

We shall not pause to complete the remaining calculations, which are elementary but laborious. We determine f_z and $\partial\psi/\partial z$ from equations (8.11),

† In mathematical terms, G_{ik} is the GREEN's tensor for the equations of equilibrium of a semi-infinite medium.

(8.15) and (8.17). Knowing $\partial\psi/\partial z$, it is easy to calculate $\partial\psi/\partial x$ and $\partial\psi/\partial y$ by integrating with respect to z and then differentiating with respect to x and y. We thus obtain all the quantities needed to calculate the displacement vector from (8.2), (8.5) and (8.7). The following are the final formulae:

$$\left.\begin{aligned} u_x &= \frac{1+\sigma}{2\pi E}\left\{\left[\frac{xz}{r^3}-\frac{(1-2\sigma)x}{r(r+z)}\right]F_z+\frac{2(1-\sigma)r+z}{r(r+z)}F_x+\right. \\ &\qquad \left.+\frac{[2r(\sigma r+z)+z^2]x}{r^3(r+z)^2}(xF_x+yF_y)\right\}, \\ u_y &= \frac{1+\sigma}{2\pi E}\left\{\left[\frac{yz}{r^3}-\frac{(1-2\sigma)y}{r(r+z)}\right]F_z+\frac{2(1-\sigma)r+z}{r(r+z)}F_y+\right. \\ &\qquad \left.+\frac{[2r(\sigma r+z)+z^2]y}{r^3(r+z)^2}(xF_x+yF_y)\right\}, \\ u_z &= \frac{1+\sigma}{2\pi E}\left\{\left[\frac{2(1-\sigma)}{r}+\frac{z^2}{r^3}\right]F_z+\left[\frac{1-2\sigma}{r(r+z)}+\frac{z}{r^3}\right](xF_x+yF_y)\right\}. \end{aligned}\right\} \quad (8.18)$$

In particular, the displacement of points on the surface of the medium is given by putting $z = 0$:

$$\left.\begin{aligned} u_x &= \frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{-\frac{(1-2\sigma)x}{r}F_z+2(1-\sigma)F_x+\frac{2\sigma x}{r^2}(xF_x+yF_y)\right\}, \\ u_y &= \frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{-\frac{(1-2\sigma)y}{r}F_z+2(1-\sigma)F_y+\frac{2\sigma y}{r^2}(xF_x+yF_y)\right\}, \\ u_z &= \frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{2(1-\sigma)F_z+(1-2\sigma)\frac{1}{r}(xF_x+yF_y)\right\}. \end{aligned}\right\} \quad (8.19)$$

PROBLEM

Determine the deformation of an infinite elastic medium when a force $\mathbf{F}$ is applied to a small region in it. †

SOLUTION. If we consider the deformation at distances r which are large compared with the dimension of the region where the force is applied, we can suppose that the force is applied at a point. The equation of equilibrium is (cf. (7.2))

$$\triangle\mathbf{u}+\frac{1}{1-2\sigma}\,\mathbf{grad}\operatorname{div}\mathbf{u} = -\frac{2(1+\sigma)}{E}\mathbf{F}\delta(\mathbf{r}), \quad (1)$$

where $\delta(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z)$, the origin being at the point where the force is applied. We seek the solution in the form $\mathbf{u} = \mathbf{u}_0+\mathbf{u}_1$, where $\mathbf{u}_0$ satisfies the Poisson-type equation

$$\triangle\mathbf{u}_0 = -\frac{2(1+\sigma)}{E}\mathbf{F}\delta(\mathbf{r}). \quad (2)$$

---

† The corresponding problem for an arbitrary infinite anisotropic medium has been solved by I. M. Lifshitz and L. N. Rozentsveig (*Zhurnal éksperimental'noĭ i teoreticheskoĭ fiziki* **17**, 783, 1947).

We then have for $\mathbf{u}_1$ the equation

$$\mathbf{grad}\ \mathrm{div}\ \mathbf{u}_1 + (1-2\sigma)\triangle \mathbf{u}_1 = -\mathbf{grad}\ \mathrm{div}\ \mathbf{u}_0. \tag{3}$$

The solution of equation (2) which vanishes at infinity is $\mathbf{u}_0 = (1+\sigma)\mathbf{F}/2\pi E r$. Taking the curl of equation (3), we have $\triangle\ \mathbf{curl}\ \mathbf{u}_1 = 0$. At infinity we must have $\mathbf{curl}\ \mathbf{u}_1 = 0$. But a function harmonic in all space and zero at infinity must be zero identically. Thus $\mathbf{curl}\ \mathbf{u}_1 = 0$, and we can therefore write $\mathbf{u}_1 = \mathbf{grad}\ \phi$. From (3) we obtain $\mathbf{grad}\ \{2(1-\sigma)\triangle\phi + \mathrm{div}\ \mathbf{u}_0\} = 0$. Hence it follows that the quantity in braces is a constant, and it must be zero at infinity; we therefore have in all space

$$\triangle\phi = -\frac{\mathrm{div}\ \mathbf{u}_0}{2(1-\sigma)} = -\frac{1+\sigma}{4\pi E(1-\sigma)}\mathbf{F}\cdot\mathbf{grad}\left(\frac{1}{r}\right).$$

If ψ is a solution of the equation $\triangle\psi = 1/r$, then

$$\phi = -\frac{1+\sigma}{4\pi E(1-\sigma)}\mathbf{F}\cdot\mathbf{grad}\ \psi.$$

Taking the solution $\psi = \frac{1}{2}r$, which has no singularities, we obtain

$$\mathbf{u}_1 = \mathbf{grad}\ \phi = \frac{1+\sigma}{8\pi E(1-\sigma)}\frac{(\mathbf{F}\cdot\mathbf{n})\mathbf{n}-\mathbf{F}}{r},$$

where $\mathbf{n}$ is a unit vector parallel to the radius vector $\mathbf{r}$. The final result is

$$\mathbf{u} = \frac{1+\sigma}{8\pi E(1-\sigma)}\cdot\frac{(3-4\sigma)\mathbf{F}+\mathbf{n}(\mathbf{n}\cdot\mathbf{F})}{r}.$$

On putting this formula into the form (8.13) we obtain the GREEN's tensor for the equations of equilibrium of an infinite isotropic medium:†

$$\begin{aligned} G_{ik} &= \frac{1+\sigma}{8\pi E(1-\sigma)}[(3-4\sigma)\delta_{ik}+n_i n_k]\frac{1}{r} \\ &= \frac{1}{4\pi\mu}\left[\frac{\delta_{ik}}{r} - \frac{1}{4(1-\sigma)}\frac{\partial^2 r}{\partial x_i \partial x_k}\right]. \end{aligned}$$

§9. Solid bodies in contact

Let two solid bodies be in contact at a point which is not a singular point on either surface. Fig. 1a shows a cross-section of the two surfaces near the point of contact O. The surfaces have a common tangent plane at O, which we take as the xy-plane. We regard the positive z-direction as being into either body (i.e. in opposite directions for the two bodies) and denote the corresponding co-ordinates by z and z'.

---

† The fact that the components of the tensor G_{ik} are first-order homogeneous functions of the co-ordinates x, y, z is evident from arguments of homogeneity applied to the form of equation (1), where the left-hand side is a linear combination of the second derivatives of the components of the vector $\mathbf{u}$, and the right-hand side is a third-order homogeneous function ($\delta(a\mathbf{r}) = a^{-3}\delta(\mathbf{r})$). This property remains valid in the general case of an arbitrary anisotropic medium.

Near a point of ordinary contact with the xy-plane, the equation of the surface can be written

$$z = \kappa_{\alpha\beta}x_{\alpha}x_{\beta}, \tag{9.1}$$

where summation is understood over the values 1, 2, of the repeated suffixes α, β ($x_1 = x$, $x_2 = y$), and $\kappa_{\alpha\beta}$ is a symmetrical tensor of rank two, which characterises the curvature of the surface: the principal values of the tensor $\kappa_{\alpha\beta}$ are $1/2R_1$ and $1/2R_2$, where R_1 and R_2 are the principal radii of curvature of the surface at the point of contact. A similar relation for the surface of the other body near the point of contact can be written

$$z' = \kappa'_{\alpha\beta}x_{\alpha}x_{\beta}. \tag{9.2}$$

Let us now assume that the two bodies are pressed together by applied forces, and approach a short distance h.† Then a deformation occurs near the original point of contact, and the two bodies will be in contact over a small but finite portion of their surfaces. Let u_z and u'_z be the components (along the z and z' axes respectively) of the corresponding displacement vectors for points on the surfaces of the two bodies. The broken lines

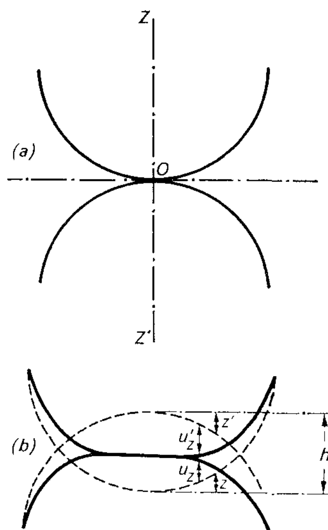


Fig. 1

in Fig. 1b show the surfaces as they would be in the absence of any deformation, while the continuous lines show the surfaces of the deformed bodies; the letters z and z' denote the distances given by equations (9.1) and (9.2). It is seen at once from the figure that the equation

$$(z+u_z)+(z'+u'_z) = h,$$

or

$$(\kappa_{\alpha\beta}+\kappa'_{\alpha\beta})x_{\alpha}x_{\beta}+u_z+u'_z = h, \tag{9.3}$$

† This *contact problem* in the theory of elasticity was first solved by H. Hertz.

holds everywhere in the region of contact. At points outside the region of contact, we have

$$z+z'+u_z+u'_z < h.$$

We choose the x and y axes to be the principal axes of the tensor $\kappa_{\alpha\beta}+\kappa'_{\alpha\beta}$. Denoting the principal values of this tensor by A and B,† we can rewrite equation (9.3) as

$$Ax^2+By^2+u_z+u'_z = h. \tag{9.4}$$

We denote by $P_z(x, y)$ the pressure between the two deformed bodies at points in the region of contact; outside this region, of course $P_z = 0$. To determine the relation between P_z and the displacements u_z, u'_z, we can with sufficient accuracy regard the surfaces as plane and use the formulae obtained in §8. According to the third of formulae (8.19) and (8.14), the displacement u_z under the action of normal forces $P_z(x, y)$ is given by

$$\begin{aligned} u_z &= \frac{1-\sigma^2}{\pi E}\iint\frac{P_z(x', y')}{r}\,dx'\,dy', \\ u'_z &= \frac{1-\sigma'^2}{\pi E'}\iint\frac{P_z(x', y')}{r}\,dx'\,dy', \end{aligned} \tag{9.5}$$

where σ, σ' and E, E' are the POISSON's ratios and the YOUNG's moduli of the two bodies. Since $P_z = 0$ outside the region of contact, the integration extends only over this region. It may be noted that, from these formulae, the ratio u_z/u'_z is constant:

$$u_z/u'_z = (1-\sigma^2)E'/(1-\sigma'^2)E. \tag{9.6}$$

The relations (9.4) and (9.6) together give the displacements u_z, u'_z at every point of the region of contact (although (9.5) and (9.6), of course, relate to points outside that region also).

Substituting the expressions (9.5) in (9.4), we obtain

$$\frac{1}{\pi}\left(\frac{1-\sigma^2}{E}+\frac{1-\sigma'^2}{E'}\right)\iint\frac{P_z(x', y')}{r}\,dx'\,dy' = h-Ax^2-By^2. \tag{9.7}$$

† The quantities A and B are related to the radii of curvature R_1, R_2 and R'_1, R'_2 by

$$\begin{aligned} 2(A+B) &= \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R'_1}+\frac{1}{R'_2}, \\ 4(A-B)^2 &= \left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2+\left(\frac{1}{R'_1}-\frac{1}{R'_2}\right)^2+ \\ &\quad +2\cos 2\phi\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R'_1}-\frac{1}{R'_2}\right), \end{aligned}$$

where ϕ is the angle between the normal sections whose radii of curvature are R_1 and R'_1.

The radii of curvature are regarded as positive if the centre of curvature lies within the body concerned, and negative in the contrary case.

This integral equation determines the distribution of the pressure P_z over the region of contact. Its solution can be found by analogy with the following results of potential theory. The idea of using this analogy arises as follows: firstly, the integral on the left-hand side of equation (9.7) is of a type commonly found in potential theory, where such integrals give the potential of a charge distribution; secondly, the potential inside a uniformly charged ellipsoid is a quadratic function of the co-ordinates.

If the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2 = 1$ is uniformly charged (with volume charge density ρ), the potential in the ellipsoid is given by

$$\phi(x, y, z)$$

$$= \pi\rho abc\int_0^\infty\left\{1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}-\frac{z^2}{c^2+\xi}\right\}\frac{d\xi}{\sqrt{\{(a^2+\xi)(b^2+\xi)(c^2+\xi)\}}}.$$

In the limiting case of an ellipsoid which is very much flattened in the z-direction ($c \to 0$), we have

$$\phi(x, y) = \pi\rho abc\int_0^\infty\left\{1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right\}\frac{d\xi}{\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}};$$

in passing to the limit $c \to 0$ we must, of course, put $z = 0$ for points inside the ellipsoid. The potential $\phi(x, y, z)$ can also be written as

$$\phi(x, y, z) = \iiint\frac{\rho\, dx'\, dy'\, dz'}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}},$$

where the integration is over the volume of the ellipsoid. In passing to the limit $c \to 0$, we must put $z = z' = 0$ in the radicand; integrating over z' between the limits

$$\pm c\sqrt{\{1-(x'^2/a^2)-(y'^2/b^2)\}},$$

we obtain

$$\phi(x, y) = 2\rho c\iint\frac{dx'\, dy'}{r}\sqrt{\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)},$$

where

$$r = \sqrt{\{(x-x')^2+(y-y')^2\}},$$

and the integration is over the area inside the ellipse

$$x'^2/a^2+y'^2/b^2 = 1.$$

Equating the two expressions for $\phi(x, y)$, we obtain the identity

$$\iint \frac{\mathrm{d}x'\,\mathrm{d}y'}{r}\sqrt{\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)}$$

$$= \tfrac{1}{2}\pi ab\int_0^\infty \left(1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right)\frac{\mathrm{d}\xi}{\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}}. \tag{9.8}$$

Comparing this relation with equation (9.7), we see that the right-hand sides are quadratic functions of x and y of the same form, and the left-hand sides are integrals of the same form. We can therefore deduce immediately that the region of contact (i.e. the region of integration in (9.7)) is bounded by an ellipse of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \tag{9.9}$$

and that the function $P_z(x, y)$ must be of the form

$$P_z(x, y) = \text{constant}\times\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}.$$

Taking the constant such that the integral $\iint P_z\,\mathrm{d}x\,\mathrm{d}y$ over the region of contact is equal to the given total force F which moves the bodies together, we obtain

$$P_z(x, y) = \frac{3F}{2\pi ab}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}. \tag{9.10}$$

This formula gives the distribution of pressure over the area of the region of contact. It may be pointed out that the pressure at the centre of this region is $\frac{3}{2}$ times the mean pressure $F/\pi ab$.

Substituting (9.10) in equation (9.7) and replacing the resulting integral in accordance with (9.8), we obtain

$$\frac{FD}{\pi}\int_0^\infty \left(1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right)\mathrm{d}\xi/\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}$$

$$= h-Ax^2-By^2,$$

where

$$D = \frac{3}{4}\left(\frac{1-\sigma^2}{E}+\frac{1-\sigma'^2}{E'}\right).$$

This equation must hold identically for all values of x and y inside the

ellipse (9.9); the coefficients of x and y and the free terms must therefore be respectively equal on each side. Hence we find

$$h = \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}}, \tag{9.11}$$

$$A = \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{(a^2+\xi)\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}},$$
$$B = \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{(b^2+\xi)\sqrt{\{(a^2+\xi)(b^2+\xi)\xi\}}}. \tag{9.12}$$

Equations (9.12) determine the semi-axes a and b of the region of contact from the given force F (A and B being known for given bodies). The relation (9.11) then gives the distance of approach h as a function of the force F. The right-hand sides of these equations involve elliptic integrals.

Thus the problem of bodies in contact can be regarded as completely solved. The form of the surfaces (i.e. the displacements u_z, u'_z) outside the region of contact is determined by the same formulae (9.5) and (9.10); the values of the integrals can be found immediately from the analogy with the potential outside a charged ellipsoid. Finally, the formulae of §8 enable us to find also the deformation at various points in the bodies (but only, of course, at distances small compared with the dimensions of the bodies).

Let us apply these formulae to the case of contact between two spheres of radii R and R'. Here $A = B = 1/2R+1/2R'$. It is clear from symmetry that $a = b$, i.e. the region of contact is a circle. From (9.12) we find the radius a of this circle to be

$$a = F^{1/3}\{DRR'/(R+R')\}^{1/3}. \tag{9.13}$$

h is in this case the difference between the sum $R+R'$ and the distance between the centres of the spheres. From (9.10) we obtain the following relation between F and h:

$$h = F^{2/3}\left[D^2\left(\frac{1}{R}+\frac{1}{R'}\right)\right]^{1/3}. \tag{9.14}$$

It should be noticed that h is proportional to $F^{2/3}$; conversely, the force F varies as $h^{3/2}$. We can write down also the potential energy U of the spheres in contact. Since $-F = -\partial U/\partial h$, we have

$$U = h^{5/2}\frac{2}{5D}\sqrt{\frac{RR'}{R+R'}}. \tag{9.15}$$

Finally, it may be mentioned that a relation of the form h = constant $\times F^{2/3}$, or F = constant $\times h^{3/2}$, holds not only for spheres but also for other finite

bodies in contact. This is easily seen from similarity arguments. If we make the substitution

$$a^2 \to \alpha a^2, \qquad b^2 \to \alpha b^2, \qquad F \to \alpha^{3/2} F,$$

where α is an arbitrary constant, equations (9.12) remain unchanged. In equation (9.11), the right-hand side is multiplied by α, and so h must be replaced by αh if this equation is to remain unchanged. Hence it follows that F must be proportional to $h^{3/2}$.

PROBLEMS

PROBLEM 1. Determine the time for which two colliding elastic spheres remain in contact.

SOLUTION. In a system of co-ordinates in which the centre of mass of the two spheres is at rest, the energy before the collision is equal to the kinetic energy of the relative motion $\frac{1}{2}\mu v^2$, where v is the relative velocity of the colliding spheres and $\mu = m_1 m_2/(m_1+m_2)$ their reduced mass. During the collision, the total energy is the sum of the kinetic energy, which may be written $\frac{1}{2}\mu \dot{h}^2$, and the potential energy (9.15). By the law of conservation of energy we have

$$\mu\left(\frac{dh}{dt}\right)^2 + kh^{5/2} = \mu v^2, \qquad k = \frac{4}{5D}\sqrt{\frac{RR'}{R+R'}}.$$

The maximum approach h_0 of the spheres corresponds to the time when their relative velocity $\dot{h} = 0$, and is $h_0 = (\mu/k)^{2/5} v^{4/5}$.

The time τ during which the collision takes place (i.e. h varies from 0 to h_0 and back) is

$$\tau = 2\int_0^{h_0} \frac{dh}{\sqrt{(v^2 - kh^{5/2}/\mu)}} = 2\left(\frac{\mu^2}{k^2 v}\right)^{1/5} \int_0^1 \frac{dx}{\sqrt{(1-x^{2/5})}},$$

or

$$\tau = \frac{4\sqrt{\pi}\,\Gamma(2/5)}{5\Gamma(9/10)}\left(\frac{\mu^2}{k^2 v}\right)^{1/5} = 2{\cdot}94\left(\frac{\mu^2}{k^2 v}\right)^{1/5}.$$

By using the statical formulae obtained in the text to solve this problem, we have neglected elastic oscillations of the spheres resulting from the collision. If this is legitimate, the velocity v must be small compared with the velocity of sound. In practice, however, the validity of the theory is limited by the still more stringent requirement that the resulting deformations should not exceed the elastic limit of the substance.

PROBLEM 2. Determine the dimensions of the region of contact and the pressure distribution when two cylinders are pressed together along a generator.

SOLUTION. In this case the region of contact is a narrow strip along the length of the cylinders. Its width $2a$ and the pressure distribution across it can be found from the formulae in the text by going to the limit $b/a \to \infty$. The pressure distribution will be of the form $P_z(x) = \text{constant} \times \sqrt{(1 - x^2/a^2)}$, where x is the co-ordinate across the strip; normalising the pressure to give a force F per unit length, we obtain

$$P_z(x) = \frac{2F}{\pi a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)}.$$

Substituting this expression in (9.7) and effecting the integration by means of (9.8), we have

$$A = \frac{4DF}{3\pi}\int_0^\infty \frac{d\xi}{(a^2+\xi)^{3/2}\xi} = \frac{8DF}{3\pi a^2}.$$

One of the radii of curvature of a cylindrical surface is infinite, and the other is the radius of the cylinder; in this case, therefore, $A = 1/2R+1/2R'$, $B = 0$. We have finally for the width of the region of contact

$$a = \sqrt{\left(\frac{16DF}{3\pi}\cdot\frac{RR'}{R+R'}\right)}.$$

§10. The elastic properties of crystals

The change in the free energy in isothermal compression of a crystal is, as with isotropic bodies, a quadratic function of the strain tensor. Unlike what happens for isotropic bodies, however, this function contains not just two coefficients, but a larger number of them. The general form of the free energy of a deformed crystal is

$$F = \tfrac{1}{2}\lambda_{iklm}u_{ik}u_{lm}, \tag{10.1}$$

where λ_{iklm} is a tensor of rank four, called the *elastic modulus tensor*. Since the strain tensor is symmetrical, the product $u_{ik}u_{lm}$ is unchanged when the suffixes i, k, or l, m, or i, l and k, m, are interchanged. Hence we see that the tensor λ_{iklm} can be defined so that it has the same symmetry properties:

$$\lambda_{iklm} = \lambda_{kilm} = \lambda_{ikml} = \lambda_{lmik}. \tag{10.2}$$

A simple calculation shows that the number of different components of a tensor of rank four having these symmetry properties is in general 21.

In accordance with the expression (10.1) for the free energy, the stress tensor for a crystal is given in terms of the strain tensor by

$$\sigma_{ik} = \partial F/\partial u_{ik} = \lambda_{iklm}u_{lm}; \tag{10.3}$$

cf. also the last footnote to this section.

If the crystal possesses symmetry, relations exist between the various components of the tensor λ_{iklm}, so that the number of independent components is less than 21.

We shall discuss these relations for each possible type of macroscopic symmetry of crystals, i.e. for each of the crystal classes, dividing these into the corresponding crystal systems.

(1) *Triclinic system.* Triclinic symmetry (classes C_1 and C_i) does not place any restrictions on the components of the tensor λ_{iklm}, and the system of co-ordinates may be chosen arbitrarily as regards the symmetry. All the 21 moduli of elasticity are non-zero and independent. However, the arbitrariness of the choice of co-ordinate system enables us to impose additional conditions on the components of the tensor λ_{iklm}. Since the orientation of the co-ordinate system relative to the body is defined by three quantities (angles of rotation), there can be three such conditions; for example, three of the components may

be taken as zero. Then the independent quantities which describe the elastic properties of the crystal will be 18 non-zero moduli and 3 angles defining the orientation of the axes in the crystal.

(2) *Monoclinic system.* Let us consider the class C_s; we take a co-ordinate system with the xy-plane as the plane of symmetry. On reflection in this plane, the co-ordinates undergo the transformation $x \to x$, $y \to y$, $z \to -z$. The components of a tensor are transformed as the products of the corresponding co-ordinates. It is therefore clear that, in the transformation mentioned, all components λ_{iklm} whose suffixes include z an odd number of times (1 or 3) will change sign, while the other components will remain unchanged. By the symmetry of the crystal, however, all quantities characterising its properties (including all components λ_{iklm}) must remain unchanged on reflection in the plane of symmetry. Hence it is evident that all components with an odd number of suffixes z must be zero. Accordingly, the general expression for the elastic free energy of a crystal belonging to the monoclinic system is

$$F = \tfrac{1}{2}\lambda_{xxxx}u_{xx}^2 + \tfrac{1}{2}\lambda_{yyyy}u_{yy}^2 + \tfrac{1}{2}\lambda_{zzzz}u_{zz}^2 + \lambda_{xxyy}u_{xx}u_{yy} + \lambda_{xxzz}u_{xx}u_{zz} + \lambda_{yyzz}u_{yy}u_{zz} + 2\lambda_{xyxy}u_{xy}^2 + 2\lambda_{xzxz}u_{xz}^2 + 2\lambda_{yzyz}u_{yz}^2 + 2\lambda_{xxxy}u_{xx}u_{xy} + 2\lambda_{yyyx}u_{yy}u_{yx} + 2\lambda_{xyzz}u_{xy}u_{zz} + 4\lambda_{xzyz}u_{xz}u_{yz}. \tag{10.4}$$

This contains 13 independent coefficients. A similar expression is obtained for the class C_2, and also for the class C_{2h}, which contains both symmetry elements (C_2 and σ_h). In the argument given, however, the direction of only one co-ordinate axis (that of z) is fixed; those of x and y can have arbitrary directions in the perpendicular plane. This arbitrariness can be used to make one coefficient, say λ_{xyzz}, vanish by a suitable choice of axes. Then the 13 quantities which describe the elastic properties of the crystal will be 12 non-zero moduli and one angle defining the orientation of the axes in the xy-plane.

(3) *Orthorhombic system.* In all the classes of this system (C_{2v}, D_2, D_{2h}) the choice of co-ordinate axes is determined by the symmetry, and the expression obtained for the free energy is the same for each class.

Let us consider, for example, the class D_{2h}; we take the three planes of symmetry as the co-ordinate planes. Reflections in each of these planes are transformations in which one co-ordinate changes sign and the other two remain unchanged. It is evident therefore that the only non-zero components λ_{iklm} are those whose suffixes contain each of x, y, z an even number of times; the other components would have to change sign on reflection in some plane of symmetry. Thus the general expression for the free energy in the orthorhombic system is

$$F = \tfrac{1}{2}\lambda_{xxxx}u_{xx}^2 + \tfrac{1}{2}\lambda_{yyyy}u_{yy}^2 + \tfrac{1}{2}\lambda_{zzzz}u_{zz}^2 + \lambda_{xxyy}u_{xx}u_{yy} + \lambda_{xxzz}u_{xx}u_{zz} + \lambda_{yyzz}u_{yy}u_{zz} + 2\lambda_{xyxy}u_{xy}^2 + 2\lambda_{xzxz}u_{xz}^2 + 2\lambda_{yzyz}u_{yz}^2. \tag{10.5}$$

It contains nine moduli of elasticity.

(4) *Tetragonal system.* Let us consider the class C_{4v}; we take the axis C_4 as the z-axis, and the x and y axes perpendicular to two of the vertical planes of symmetry. Reflections in these two planes signify transformations

$$x \to -x, \quad y \to y, \quad z \to z$$

and

$$x \to x, \quad y \to -y, \quad z \to z;$$

all components λ_{iklm} with an odd number of like suffixes therefore vanish. Furthermore, a rotation through an angle $\frac{1}{4}\pi$ about the axis C_4 is the transformation

$$x \to y, \quad y \to -x, \quad z \to z.$$

Hence we have

$$\lambda_{xxxx} = \lambda_{yyyy}, \quad \lambda_{xxzz} = \lambda_{yyzz}, \quad \lambda_{xzxz} = \lambda_{yzyz}.$$

The remaining transformations in the class C_{4v} do not give any further conditions. Thus the free energy of crystals in the tetragonal system is

$$F = \tfrac{1}{2}\lambda_{xxxx}(u_{xx}^2+u_{yy}^2)+\tfrac{1}{2}\lambda_{zzzz}u_{zz}^2+\lambda_{xxzz}(u_{xx}u_{zz}+u_{yy}u_{zz})+ \\ +\lambda_{xxyy}u_{xx}u_{yy}+2\lambda_{xyxy}u_{xy}^2+2\lambda_{xzxz}(u_{xz}^2+u_{yz}^2). \qquad (10.6)$$

It contains six moduli of elasticity.

A similar result is obtained for those other classes of the tetragonal system where the natural choice of the co-ordinate axes is determined by symmetry (D_{2d}, D_4, D_{4h}). In the classes C_4, S_4, C_{4h}, on the other hand, only the choice of the z-axis is unique (along the axis C_4 or S_4). The requirements of symmetry then allow a further component $\lambda_{xxxy} = -\lambda_{yyyx}$ in addition to those which appear in (10.6). These components may be made to vanish by suitably choosing the directions of the x and y axes, and F then reduces to the form (10.6).

(5) *Rhombohedral system.* Let us consider the class C_{3v}; we take the third order axis as the z-axis, and the y-axis perpendicular to one of the vertical planes of symmetry. In order to find the restrictions imposed on the components of the tensor λ_{iklm} by the presence of the axis C_3, it is convenient to make a formal transformation using the complex co-ordinates $\xi = x+iy$, $\eta = x-iy$, the z co-ordinate remaining unchanged. We transform the tensor λ_{iklm} to the new co-ordinate system also, so that its suffixes take the values ξ, η, z. It is easy to see that, in a rotation through $2\pi/3$ about the axis C_3, the new co-ordinates undergo the transformation $\xi \to \xi e^{2\pi i/3}$, $\eta \to \eta e^{-2\pi i/3}$, $z \to z$. By symmetry only those components λ_{iklm} which are unchanged by this transformation can be different from zero. These components are evidently the ones whose suffixes contain ξ three times, or η three times (since $(e^{2\pi i/3})^3 = e^{2\pi i} = 1$), or ξ and η the same number of times (since $e^{2\pi i/3}e^{-2\pi i/3} = 1$), i.e. λ_{zzzz}, $\lambda_{\xi\eta\xi\eta}$, $\lambda_{\xi\xi\eta\eta}$, $\lambda_{\xi\eta zz}$, $\lambda_{\xi z\eta z}$, $\lambda_{\xi\xi\xi z}$, $\lambda_{\eta\eta\eta z}$.

Furthermore, a reflection in the symmetry plane perpendicular to the y-axis gives the transformation $x \to x$, $y \to -y$, $z \to z$, or $\xi \to \eta$, $\eta \to \xi$. Since $\lambda_{\xi\xi\xi z}$ becomes $\lambda_{\eta\eta\eta z}$ in this transformation, these two components must be equal. Thus crystals of the rhombohedral system have only six moduli of elasticity. In order to obtain an expression for the free energy, we must form the sum $\frac{1}{2}\lambda_{iklm}u_{ik}u_{lm}$, in which the suffixes take the values ξ, η, z; since F is to be expressed in terms of the components of the strain tensor in the co-ordinates x, y, z, we must express in terms of these the components in the co-ordinates ξ, η, z. This is easily done by using the fact that the components of the tensor u_{ik} transform as the products of the corresponding co-ordinates. For example, since

$$\xi^2 = (x+iy)^2 = x^2-y^2+2ixy,$$

it follows that

$$u_{\xi\xi} = u_{xx}-u_{yy}+2iu_{xy}.$$

Consequently, the expression for F is found to be

$$F = \tfrac{1}{2}\lambda_{zzzz}u_{zz}^2+2\lambda_{\xi\eta\xi\eta}(u_{xx}+u_{yy})^2+\lambda_{\xi\xi\eta\eta}\{(u_{xx}-u_{yy})^2+4u_{xy}^2\}+$$
$$+2\lambda_{\xi\eta zz}(u_{xx}+u_{yy})u_{zz}+4\lambda_{\xi z\eta z}(u_{xz}^2+u_{yz}^2)+4\lambda_{\xi\xi\xi z}\{(u_{xx}-u_{yy})u_{xz}-2u_{xy}u_{yz}\}. \tag{10.7}$$

This contains 6 independent coefficients. A similar result is obtained for the classes $\boldsymbol{D}_3$ and $\boldsymbol{D}_{3d}$, but in the classes $\boldsymbol{C}_3$ and $\boldsymbol{S}_6$, where the choice of the x and y axes remains arbitrary, requirements of symmetry allow also a non-zero value of the difference $\lambda_{\xi\xi\xi z}-\lambda_{\eta\eta\eta z}$. This, however, can be made to vanish by a suitable choice of the x and y axes.

(6) *Hexagonal system.* Let us consider the class $\boldsymbol{C}_6$; we take the sixth-order axis as the z-axis, and again use the co-ordinates $\xi = x+iy$, $\eta = x-iy$. In a rotation through an angle $\frac{1}{3}\pi$ about the z-axis, the co-ordinates ξ, η undergo the transformation $\xi \to \xi e^{\pi i/3}$, $\eta \to \eta e^{-\pi i/3}$. Hence we see that only those components λ_{iklm} are non-zero which contain the same number of suffixes ξ and η. These are λ_{zzzz}, $\lambda_{\xi\eta\xi\eta}$, $\lambda_{\xi\xi\eta\eta}$, $\lambda_{\xi\eta zz}$, $\lambda_{\xi z\eta z}$. Other symmetry elements in the hexagonal system give no further restrictions. There are therefore only five moduli of elasticity. The free energy is

$$F = \tfrac{1}{2}\lambda_{zzzz}u_{zz}^2+2\lambda_{\xi\eta\xi\eta}(u_{xx}+u_{yy})^2+\lambda_{\xi\xi\eta\eta}[(u_{xx}-u_{yy})^2+4u_{xy}^2]+$$
$$+2\lambda_{\xi\eta zz}u_{zz}(u_{xx}+u_{yy})+4\lambda_{\xi z\eta z}(u_{xz}^2+u_{yz}^2). \tag{10.8}$$

It should be noticed that a deformation in the xy-plane (for which u_{xx}, u_{yy} and u_{xy} are non-zero) is determined by only two moduli of elasticity, as for an isotropic body; that is, the elastic properties of a hexagonal crystal are isotropic in the plane perpendicular to the sixth-order axis.

For this reason the choice of axis directions in this plane is unimportant and does not affect the form of F. The expression (10.8) therefore applies to all classes of the hexagonal system.

(7) *Cubic system.* We take the axes along the three fourth-order axes of the cubic system. Since there is tetragonal symmetry (with the fourth-order axis in the z-direction), the number of different components of the tensor λ_{iklm} is limited to at most the following six: λ_{xxxx}, λ_{zzzz}, λ_{xxzz}, λ_{xxyy}, λ_{xyxy}, λ_{xzxz}. Rotations through $\frac{1}{2}\pi$ about the x and y axes give respectively the transformations $x \to x$, $y \to -z$, $z \to y$, and $x \to z$, $y \to y$, $z \to -x$. The components listed are therefore equal in successive pairs. Thus there remain only three different moduli of elasticity. The free energy of crystals of the cubic system is

$$F = \tfrac{1}{2}\lambda_{xxxx}(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + \lambda_{xxyy}(u_{xx}u_{yy} + u_{xx}u_{zz} + u_{yy}u_{zz}) + + 2\lambda_{xyxy}(u_{xy}^2 + u_{xz}^2 + u_{yz}^2). \quad (10.9)$$

We may recapitulate the number of independent parameters (elastic moduli or angles defining the orientation of axes in the crystal) for the classes of the various systems:

Triclinic	21
Monoclinic	13
Orthorhombic	9
Tetragonal (C_4, S_4, C_{4h})	7
Tetragonal (C_{4v}, D_{2d}, D_4, D_{4h})	6
Rhombohedral (C_3, S_6)	7
Rhombohedral (C_{3v}, D_3, D_{3d})	6
Hexagonal	5
Cubic	3

The least number of non-zero moduli that is possible by suitable choice of the co-ordinate axes is the same for all the classes in each system:

Triclinic	18
Monoclinic	12
Orthorhombic	9
Tetragonal	6
Rhombohedral	6
Hexagonal	5
Cubic	3

All the above discussion relates, of course, to single crystals. Polycrystalline bodies whose component crystallites are sufficiently small may be regarded as isotropic bodies (since we are concerned with deformations in regions large compared with the dimensions of the crystallites). Like any isotropic body, a polycrystal has only two moduli of elasticity. It might be thought at first sight that these moduli could be obtained from those of the individual crystallites by simple averaging. This is not so, however. If we regard the deformation of a polycrystal as the result of a deformation of its component crystallites, it would in principle be necessary to solve the equations of

equilibrium for every crystallite, taking into account the appropriate boundary conditions at their surfaces of separation. Hence we see that the relation between the elastic properties of the whole crystal and those of its component crystallites depends on the actual form of the latter and the amount of correlation of their mutual orientations. There is therefore no general relation between the moduli of elasticity of a polycrystal and those of a single crystal of the same substance.

The moduli of an isotropic polycrystal can be calculated with fair accuracy from those of a single crystal only when the elastic properties of the single crystal are nearly isotropic.† In a first approximation, the moduli of elasticity of the polycrystal can then simply be put equal to the "isotropic part" of the moduli of the single crystal. In the next approximation, terms appear which are quadratic in the small "anisotropic part" of these moduli. It is found‡ that these correction terms are independent of the shape of the crystallites and of the correlation of their orientations, and can be calculated in a general form.

Finally, let us consider the thermal expansion of crystals. In isotropic bodies, the thermal expansion is the same in every direction, so that the strain tensor in free thermal expansion is (see §6) $u_{ik} = \frac{1}{3}\alpha(T-T_0)\delta_{ik}$, where α is the thermal expansion coefficient. In crystals, however, we must put

$$u_{ik} = \tfrac{1}{3}\alpha_{ik}(T-T_0), \tag{10.10}$$

where α_{ik} is a tensor of rank two, symmetrical in the suffixes i and k. Let us calculate the number of independent components of this tensor in crystals of the various systems. The simplest way of doing this is to use the result of tensor algebra that to every symmetrical tensor of rank two there corresponds a *tensor ellipsoid.*§ It follows at once from considerations of symmetry that, for triclinic, monoclinic and orthorhombic symmetry, the tensor ellipsoid has three axes of different length. For tetragonal, rhombohedral and hexagonal symmetry, on the other hand, we have an ellipsoid of revolution (with its axis of symmetry along the axes C_4, C_3 and C_6 respectively). Finally, for cubic symmetry the ellipsoid becomes a sphere. An ellipsoid of three axes is determined by three quantities, an ellipsoid of revolution by two, and a sphere by one (the radius). Thus the number of independent components of the tensor α_{ik} in crystals of the various systems is as follows: triclinic, monoclinic and orthorhombic, 3; tetragonal, rhombohedral and hexagonal, 2; cubic, 1.

Crystals of the first three systems are said to be *biaxial*, and those of the second three systems *uniaxial.* It should be noticed that the thermal expansion of crystals of the cubic system is determined by one quantity only, i.e. they behave in this respect as isotropic bodies.

---

† For a "nearly isotropic" cubic crystal (e.g.), the difference $\lambda_{xxxx}-\lambda_{xxyy}-2\lambda_{xyxy}$ must be small.
‡ I. M. Lifshitz and L. N. Rozentsveig, *Zhurnal éksperimental'noĭ i teoreticheskoĭ fiziki* 16, 967, 1946.
§ Determined by the equation $\alpha_{ik}x_ix_k = 1$.

PROBLEM

Determine the YOUNG's modulus of a cubic crystal as a function of direction.

SOLUTION. We take the axes of co-ordinates along the three axes of the fourth order. Let the axis of a rod cut from the crystal be in the direction of the unit vector **n**. The stress tensor σ_{ik} in the extended rod must satisfy the following conditions: when σ_{ik} is multiplied by n_i, the resulting extension force must be parallel to **n** (condition at the ends of the rod); when it is multiplied by a vector perpendicular to **n**, the result must be zero (condition on the sides of the rod). Such a tensor must be of the form $\sigma_{ik} = pn_i n_k$, where p is the extension force per unit area of the ends of the rod. Calculating the components σ_{ik} by means of the expression (10.9) for the free energy† and comparing them with the formulae $\sigma_{ik} = pn_i n_k$, we find the components of the strain tensor to be

$$u_{xx} = p\frac{(c_1+2c_2)n_x^2 - c_2}{(c_1-c_2)(c_1+2c_2)}, \qquad u_{xy} = pn_x n_y/2c_3,$$

and similarly for the remaining components. Here we have put $\lambda_{xxxx} = c_1$, $\lambda_{xxyy} = c_2$, $\lambda_{xyxy} = c_3$.

The relative longitudinal extension of the rod is $u = (\mathrm{d}l' - \mathrm{d}l)/\mathrm{d}l$, where $\mathrm{d}l'$ is given by formula (1.2) and $\mathrm{d}x_i/\mathrm{d}l = n_i$. For small deformations this gives $u = u_{ik}n_i n_k$. The YOUNG's modulus is determined by the coefficient of proportionality in $p = Eu$, and is

$$E = \left[\frac{c_1+c_2}{(c_1+2c_2)(c_1-c_2)} + \left(\frac{1}{c_3} - \frac{2}{c_1-c_2}\right)\left(n_x^2 n_y^2 + n_x^2 n_z^2 + n_y^2 n_z^2\right)\right]^{-1}.$$

E has extremum values in the directions of the edges (i.e. of the co-ordinate axes) and of the spatial diagonals of the cube.

---

† In calculating σ_{ik}, the following fact must be borne in mind. If we effect the calculation, not directly from the formulae $\sigma_{ik} = \lambda_{iklm}u_{lm}$, but by differentiation of the expression for the free energy with respect to the components of the tensor u_{ik}, the derivatives with respect to u_{ik} with $i \neq k$ give twice the values of the corresponding components σ_{ik}. This is because the expressions $\sigma_{ik} = \partial F/\partial u_{ik}$ are meaningful only as indicating that $\mathrm{d}F = \sigma_{ik}\,\mathrm{d}u_{ik}$; in the sum $\sigma_{ik}\,\mathrm{d}u_{ik}$, however, the term in the differential $\mathrm{d}u_{ik}$ for each component with $i \neq k$ of the symmetrical tensor u_{ik} appears twice.

CHAPTER II

THE EQUILIBRIUM OF RODS AND PLATES

§11. The energy of a bent plate

In this chapter we shall study some particular cases of the equilibrium of deformed bodies, and we begin with that of thin deformed plates. When we speak of a *thin plate*, we mean that its thickness is small compared with its dimensions in the other two directions. The deformations themselves are supposed small, as before. In the present case the deformation is small if the displacements of points in the plate are small compared with its thickness.

The general equations of equilibrium are considerably simplified when applied to thin plates. It is more convenient, however, not to derive these simplified equations directly from the general ones, but to calculate afresh the free energy of a bent plate and then vary that energy.

When a plate is bent, it is stretched at some points and compressed at others: on the convex side there is evidently an extension, which decreases as we penetrate into the plate, finally becoming zero, after which a gradually increasing compression is found. The plate therefore contains a *neutral surface*, on which there is no extension or compression, and on opposite sides of which the deformation has opposite signs. The neutral surface clearly lies midway through the plate.

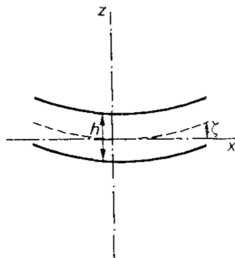


FIG. 2

We take a co-ordinate system with the origin on the neutral surface and the z-axis normal to the surface. The xy-plane is that of the undeformed plate. We denote by ζ the vertical displacement of a point on the neutral surface, i.e. its z co-ordinate (Fig. 2). The components of its displacement in the xy-plane are evidently of the second order of smallness relative to ζ, and can therefore be put equal to zero. Thus the displacement vector for points on the neutral surface is

$$u_x^{(0)} = u_y^{(0)} = 0, \qquad u_z^{(0)} = \zeta(x, y). \tag{11.1}$$

For further calculations it is necessary to note the following property of the stresses in a deformed plate. Since the plate is thin, comparatively small forces on its surface are needed to bend it. These forces are always considerably less than the internal stresses caused in the deformed plate by the extension and compression of its parts. We can therefore neglect the forces P_i in the boundary condition (2.8), leaving $\sigma_{ik}n_k = 0$. Since the plate is only slightly bent, we can suppose that the normal vector $\mathbf{n}$ is along the z-axis. Thus we must have on both surfaces of the plate $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. Since the plate is thin, however, these quantities must be small within the plate if they are zero on each surface. We therefore conclude that the components σ_{xz}, σ_{yz}, σ_{zz} are small compared with the remaining components of the stress tensor everywhere in the plate. We can therefore equate them to zero and use this condition to determine the components of the strain tensor.

By the general formulae (5.13), we have

$$\sigma_{zx} = \frac{E}{1+\sigma}u_{zx}, \qquad \sigma_{zy} = \frac{E}{1+\sigma}u_{zy},$$
$$\sigma_{zz} = \frac{E}{(1+\sigma)(1-2\sigma)}\{(1-\sigma)u_{zz}+\sigma(u_{xx}+u_{yy})\}. \tag{11.2}$$

Equating these expressions to zero, we obtain $\partial u_x/\partial z = -\partial u_z/\partial x$, $\partial u_y/\partial z = -\partial u_z/\partial y$, $u_{zz} = -\sigma(u_{xx}+u_{yy})/(1-\sigma)$. In the first two of these equations u_z can, with sufficient accuracy, be replaced by $\zeta(x, y)$: $\partial u_x/\partial z = -\partial\zeta/\partial x$, $\partial u_y/\partial z = -\partial\zeta/\partial y$, whence

$$u_x = -z\,\partial\zeta/\partial x, \qquad u_y = -z\,\partial\zeta/\partial y. \tag{11.3}$$

The constants of integration are put equal to zero in order to make

$$u_x = u_y = 0 \text{ for } z = 0.$$

Knowing u_x and u_y, we can determine all the components of the strain tensor:

$$u_{xx} = -z\partial^2\zeta/\partial x^2, \qquad u_{yy} = -z\partial^2\zeta/\partial y^2, \qquad u_{xy} = -z\partial^2\zeta/\partial x\partial y,$$
$$u_{xz} = u_{yz} = 0, \qquad u_{zz} = \frac{\sigma}{1-\sigma}z\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right). \tag{11.4}$$

We can now calculate the free energy F per unit volume of the plate, using the general formula (5.10). A simple calculation gives the expression

$$F = z^2\frac{E}{1+\sigma}\left\{\frac{1}{2(1-\sigma)}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2+\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}. \tag{11.5}$$

The total free energy of the plate is obtained by integrating over the volume. The integration over z is from $-\frac{1}{2}h$ to $+\frac{1}{2}h$, where h is the thickness of the

plate, and that over x, y is over the surface of the plate. The result is that the total free energy $F_{\text{pl}} = \int F\,\mathrm{d}V$ of a deformed plate is

$$F_{\text{pl}} = \frac{Eh^3}{24(1-\sigma^2)}\int\int\left[\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2 + \right.$$

$$\left. +2(1-\sigma)\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\right]\mathrm{d}x\,\mathrm{d}y; \qquad (11.6)$$

the element of area can with sufficient accuracy be written as $\mathrm{d}x\,\mathrm{d}y$ simply, since the deformation is small.

Having obtained the expression for the free energy, we can regard the plate as being of infinitesimal thickness, i.e. as being a geometrical surface, since we are interested only in the form which it takes under the action of the applied forces, and not in the distribution of deformations inside it. The quantity ζ is then the displacement of points on the plate, regarded as a surface, when it is bent.

§12. The equation of equilibrium for a plate

The equation of equilibrium for a plate can be derived from the condition that its free energy is a minimum. To do so, we must calculate the variation of the expression (11.6).

We divide the integral in (11.6) into two, and vary the two parts separately. The first integral can be written in the form $\int(\triangle\zeta)^2\,\mathrm{d}f$, where $\mathrm{d}f = \mathrm{d}x\,\mathrm{d}y$ is a surface element and $\triangle \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is here (and in §§13, 14) the two-dimensional Laplacian. Varying this integral, we have

$$\begin{aligned}\delta\tfrac{1}{2}\int(\triangle\zeta)^2\,\mathrm{d}f &= \int\triangle\zeta\,\triangle\delta\zeta\,\mathrm{d}f \\ &= \int\triangle\zeta\,\mathrm{div}\,\mathbf{grad}\,\delta\zeta\,\mathrm{d}f \\ &= \int\mathrm{div}\,(\triangle\zeta\,\mathbf{grad}\,\delta\zeta)\,\mathrm{d}f - \int\mathbf{grad}\,\delta\zeta\cdot\mathbf{grad}\,\triangle\zeta\,\mathrm{d}f.\end{aligned}$$

All the vector operators, of course, relate to the two-dimensional co-ordinate system (x, y). The first integral on the right can be transformed into an integral along a closed contour enclosing the plate:†

$$\begin{aligned}\int\mathrm{div}\,(\triangle\zeta\,\mathbf{grad}\,\delta\zeta)\,\mathrm{d}f &= \oint\triangle\zeta(\mathbf{n}\cdot\mathbf{grad}\,\delta\zeta)\,\mathrm{d}l \\ &= \oint\triangle\zeta\frac{\partial\delta\zeta}{\partial n}\,\mathrm{d}l,\end{aligned}$$

where $\partial/\partial n$ denotes differentiation along the outward normal to the contour.

† The transformation formula for two-dimensional integrals is exactly analogous to the one for three dimensions. The volume element $\mathrm{d}V$ is replaced by the surface element $\mathrm{d}f$ (a scalar), and the surface element $\mathbf{df}$ is replaced by a contour element $\mathrm{d}l$ multiplied by the vector $\mathbf{n}$ along the outward normal to the contour. The integral over $\mathrm{d}f$ is converted into one over $\mathrm{d}l$ by replacing the operator $\mathrm{d}f\,\partial/\partial x_i$ by $n_i\,\mathrm{d}l$. For instance, if ϕ is a scalar, we have $\int\mathbf{grad}\,\phi\,\mathrm{d}f = \oint\phi\mathbf{n}\,\mathrm{d}l$.

In the second integral we use the same transformation to obtain

$$\begin{aligned}\int \mathbf{grad}\,\delta\zeta \cdot \mathbf{grad}\,\triangle\zeta\, \mathrm{d}f &= \int \mathrm{div}\,(\delta\zeta\, \mathbf{grad}\,\triangle\zeta)\, \mathrm{d}f - \int \delta\zeta\, \triangle^2\zeta\, \mathrm{d}f \\ &= \oint \delta\zeta(\mathbf{n}\cdot\mathbf{grad}\,\triangle\zeta)\, \mathrm{d}l - \int \delta\zeta\, \triangle^2\zeta\, \mathrm{d}f \\ &= \oint \delta\zeta \frac{\partial \triangle\zeta}{\partial n}\, \mathrm{d}l - \int \delta\zeta\, \triangle^2\zeta\, \mathrm{d}f.\end{aligned}$$

Substituting these results, we find that

$$\delta\tfrac{1}{2}\int (\triangle\zeta)^2\, \mathrm{d}f = \int \delta\zeta\, \triangle^2\zeta\, \mathrm{d}f - \oint \delta\zeta \frac{\partial\triangle\zeta}{\partial n}\, \mathrm{d}l + \oint \triangle\zeta \frac{\partial\delta\zeta}{\partial n}\, \mathrm{d}l. \tag{12.1}$$

The transformation of the variation of the second integral in (11.6) is somewhat more lengthy. This transformation is conveniently effected in

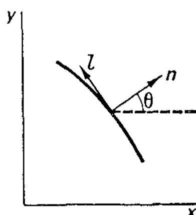


FIG. 3

components, and not in vector form. We have

$$\delta\int\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\mathrm{d}f$$

$$= \int\left\{2\frac{\partial^2\zeta}{\partial x\partial y}\frac{\partial^2\delta\zeta}{\partial x\partial y} - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\delta\zeta}{\partial y^2} - \frac{\partial^2\delta\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\mathrm{d}f.$$

The integrand can be written

$$\frac{\partial}{\partial x}\left(\frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x^2}\right),$$

i.e. as the (two-dimensional) divergence of a certain vector. The variation can therefore be written as a contour integral:

$$\begin{aligned}\delta\int\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\mathrm{d}f &= \oint \mathrm{d}l\sin\theta\left\{\frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x^2}\right\} + \\ &\quad + \oint \mathrm{d}l\cos\theta\left\{\frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial y^2}\right\},\end{aligned} \tag{12.2}$$

where θ is the angle between the x-axis and the normal to the contour (Fig. 3).

The derivatives of $\delta\zeta$ with respect to x and y are expressed in terms of its derivatives along the normal $\mathbf{n}$ and the tangent $\mathbf{l}$ to the contour:

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial n} - \sin\theta\frac{\partial}{\partial l},$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial n} + \cos\theta\frac{\partial}{\partial l}.$$

Then formula (12.2) becomes

$$\delta\int\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\mathrm{d}f$$

$$= \oint \mathrm{d}l\frac{\partial\delta\zeta}{\partial n}\left\{2\sin\theta\cos\theta\frac{\partial^2\zeta}{\partial x\partial y} - \sin^2\theta\frac{\partial^2\zeta}{\partial x^2} - \cos^2\theta\frac{\partial^2\zeta}{\partial y^2}\right\}+$$

$$+\oint \mathrm{d}l\frac{\partial\delta\zeta}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2} - \frac{\partial^2\zeta}{\partial x^2}\right) + (\cos^2\theta - \sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}.$$

The second integral may be integrated by parts. Since it is taken along a closed contour, the limits of integration are the same point, and we have simply

$$-\oint \mathrm{d}l\,\delta\zeta\frac{\partial}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2} - \frac{\partial^2\zeta}{\partial x^2}\right) + (\cos^2\theta - \sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}.$$

Collecting all the above expressions and multiplying by the coefficients shown in formula (11.6), we obtain the following final expression for the variation of the free energy:

$$\delta F_{\mathrm{pl}} = \frac{Eh^3}{12(1-\sigma^2)}\left(\int \triangle^2\zeta\delta\zeta\,\mathrm{d}f - \right.$$

$$-\oint\delta\zeta\,\mathrm{d}l\left[\frac{\partial\triangle\zeta}{\partial n} + (1-\sigma)\frac{\partial}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2} - \frac{\partial^2\zeta}{\partial x^2}\right)+\right.\right.$$

$$\left.\left.+(\cos^2\theta - \sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}\right]+$$

$$+\oint\frac{\partial\delta\zeta}{\partial n}\,\mathrm{d}l\left\{\triangle\zeta + (1-\sigma)\left(2\sin\theta\cos\theta\frac{\partial^2\zeta}{\partial x\partial y} - \right.\right.$$

$$\left.\left.\left. - \sin^2\theta\frac{\partial^2\zeta}{\partial x^2} - \cos^2\theta\frac{\partial^2\zeta}{\partial y^2}\right)\right\}\right). \qquad (12.3)$$

In order to derive from this the equation of equilibrium for the plate, we must equate to zero the sum of the variation δF and the variation δU of the potential energy of the plate due to the external forces acting on it. This latter variation is minus the work done by the external forces in deforming the

plate. Let P be the external force acting on the plate, per unit area† and normal to the surface. Then the work done by the external forces when the points on the plate are displaced a distance $\delta\zeta$ is $\int P\delta\zeta\,\mathrm{d}f$. Thus the condition for the total free energy of the plate to be a minimum is

$$\delta F_{\mathrm{pl}} - \int P\delta\zeta\,\mathrm{d}f = 0. \tag{12.4}$$

On the left-hand side of this equation we have both surface and contour integrals. The surface integral is

$$\int \left\{\frac{Eh^3}{12(1-\sigma^2)}\triangle^2\zeta - P\right\}\delta\zeta\,\mathrm{d}f.$$

The variation $\delta\zeta$ in this integral is arbitrary. The integral can therefore vanish only if the coefficient of $\delta\zeta$ is zero, i.e.

$$\frac{Eh^3}{12(1-\sigma^2)}\triangle^2\zeta - P = 0. \tag{12.5}$$

This is the equation of equilibrium for a plate bent by external forces acting on it‡.

The boundary conditions for this equation are obtained by equating to zero the contour integrals in (12.3). Here various particular cases have to be considered. Let us suppose that part of the edge of the plate is free, i.e. no external forces act on it. Then the variations $\delta\zeta$ and $\delta\partial\zeta/\partial n$ on this part of the edge are arbitrary, and their coefficients in the contour integrals must be zero. This gives the equations

$$-\frac{\partial\triangle\zeta}{\partial n} + (1-\sigma)\frac{\partial}{\partial l}\left\{\cos\theta\sin\theta\left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2}\right) + \right.$$
$$\left. + (\sin^2\theta - \cos^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\} = 0, \tag{12.6}$$

$$\triangle\zeta + (1-\sigma)\left\{2\sin\theta\cos\theta\frac{\partial^2\zeta}{\partial x\partial y} - \sin^2\theta\frac{\partial^2\zeta}{\partial x^2} - \cos^2\theta\frac{\partial^2\zeta}{\partial y^2}\right\} = 0, \tag{12.7}$$

which must hold at all free points on the edge of the plate.

The boundary conditions (12.6) and (12.7) are very complex. Considerable simplifications occur when the edge of the plate is clamped or supported. If it is clamped (Fig. 4a), no vertical displacement is possible, and moreover no

---

† The force P may be the result of body forces (e.g. the force of gravity), and is then equal to the integral of the body force over the thickness of the plate.

‡ The coefficient $D = Eh^3/12(1-\sigma^2)$ in this equation is called the *flexural rigidity* or *cylindrical rigidity* of the plate.

bending is possible at the edge. The angle through which a given part of the edge turns from its initial position is (for small displacements ζ) the derivative $\partial\zeta/\partial n$. Thus the variations $\delta\zeta$ and $\delta\partial\zeta/\partial n$ must be zero at clamped edges, so that the contour integrals in (12.3) are zero identically. The boundary conditions have in this case the simple form

$$\zeta = 0, \qquad \partial\zeta/\partial n = 0. \tag{12.8}$$

The first of these expresses the fact that the edge of the plate undergoes no vertical displacement in the deformation, and the second that it remains horizontal.

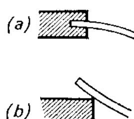


Fig. 4

It is easy to determine the reaction forces on a plate at a point where it is clamped. These are equal and opposite to the forces exerted by the plate on its support. As we know from mechanics, the force in any direction is equal to the space derivative, in that direction, of the energy. In particular, the force exerted by the plate on its support is given by minus the derivative of the energy with respect to the displacement ζ of the edge of the plate, and the reaction force by this derivative itself. The derivative in question, however, is just the coefficient of $\delta\zeta$ in the second integral in (12.3). Thus the reaction force per unit length is equal to the expression on the left of equation (12.6) (which, of course, is not now zero), multiplied by $Eh^3/12(1-\sigma^2)$.

Similarly, the moment of the reaction forces is given by the expression on the left of equation (12.7), multiplied by the same factor. This follows at once from the result of mechanics that the moment of the force is equal to the derivative of the energy with respect to the angle through which the body turns. This angle is $\partial\zeta/\partial n$, so that the corresponding moment is given by the coefficient of $\partial\delta\zeta/\partial n$ in the third integral in (12.3). Both these expressions (that for the force and that for the moment) can be very much simplified by virtue of the conditions (12.8). Since ζ and $\partial\zeta/\partial n$ are zero everywhere on the edge of the plate, their tangential derivatives of all orders are zero also. Using this and converting the derivatives with respect to x and y in (12.6) and (12.7) into those in the directions of **n** and **l**, we obtain the following simple expressions for the reaction force F and the reaction moment M:

$$F = -\frac{Eh^3}{12(1-\sigma^2)}\left[\frac{\partial^3\zeta}{\partial n^3}+\frac{\mathrm{d}\theta}{\mathrm{d}l}\frac{\partial^2\zeta}{\partial n^2}\right], \tag{12.9}$$

$$M = \frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^2\zeta}{\partial n^2}. \tag{12.10}$$

Another important case is that where the plate is supported (Fig. 4b), i.e. the edge rests on a fixed support, but is not clamped to it. In this case there is again no vertical displacement at the edge of the plate (i.e. on the line where it rests on the support), but its direction can vary. Accordingly, we have in (12.3) $\delta\zeta = 0$ in the contour integral, but $\partial\delta\zeta/\partial n \neq 0$. Hence only the condition (12.7) remains valid, and not (12.6). The expression on the left of (12.6) gives as before the reaction force at the points where the plate is supported; the moment of this force is zero in equilibrium. The boundary condition (12.7) can be simplified by converting to the derivatives in the direction of **n** and **l** and using the fact that, since $\zeta = 0$ everywhere on the edge, the derivatives $\partial\zeta/\partial l$ and $\partial^2\zeta/\partial l^2$ are also zero. We then have the boundary conditions in the form

$$\zeta = 0, \qquad \frac{\partial^2\zeta}{\partial n^2}+\sigma\frac{\mathrm{d}\theta}{\mathrm{d}l}\frac{\partial\zeta}{\partial n} = 0. \tag{12.11}$$

PROBLEMS

PROBLEM 1. Determine the deflection of a circular plate (of radius R) with clamped edges, placed horizontally in a gravitational field.

SOLUTION. We take polar co-ordinates, with the origin at the centre of the plate. The force on unit area of the surface of the plate is $P = \rho hg$. Equation (12.5) becomes $\triangle^2\zeta = 64\beta$, where $\beta = 3\rho g(1-\sigma^2)/16h^2E$; positive values of ζ correspond to displacements downward. Since ζ is a function of r only, we can put $\triangle = r^{-1}\,\mathrm{d}(r\mathrm{d}/\mathrm{d}r)/\mathrm{d}r$. The general integral is $\zeta = \beta r^4 + ar^2 + b + cr^2\log(r/R) + d\log(r/R)$. In the case in question we must put $d = 0$, since $\log(r/R)$ becomes infinite at $r = 0$, and $c = 0$, since this term gives a singularity in $\triangle\zeta$ at $r = 0$ (corresponding to a force applied at the centre of the plate; see Problem 3). The constants a and b are determined from the boundary conditions $\zeta = 0$, $\mathrm{d}\zeta/\mathrm{d}r = 0$ for $r = R$. The result is $\zeta = \beta(R^2-r^2)^2$.

PROBLEM 2. The same as Problem 1, but for a plate with supported edges.

SOLUTION. The boundary conditions (12.11) for a circular plate are

$$\zeta = 0, \qquad \frac{\mathrm{d}^2\zeta}{\mathrm{d}r^2}+\frac{\sigma}{r}\frac{\mathrm{d}\zeta}{\mathrm{d}r} = 0.$$

The solution is similar to that of Problem 1, and the result is

$$\zeta = \beta(R^2-r^2)\left(\frac{5+\sigma}{1+\sigma}R^2-r^2\right).$$

PROBLEM 3. Determine the deflection of a circular plate with clamped edges when a force f is applied to its centre.

SOLUTION. We have $\triangle^2\zeta = 0$ everywhere except at the origin. Integration gives

$$\zeta = ar^2+b+cr^2\log(r/R),$$

the $\log r$ term again being omitted. The total force on the plate is equal to the force f at its

centre. The integral of $\triangle^2\zeta$ over the surface of the plate must therefore be:

$$2\pi\int_0^R r\triangle^2\zeta\,\mathrm{d}r = \frac{12(1-\sigma^2)}{Eh^3}f.$$

Hence $c = 3(1-\sigma^2)f/2\pi Eh^3$. The constants a and b are determined from the boundary conditions. The result is

$$\zeta = \frac{3f(1-\sigma^2)}{2\pi Eh^3}[\tfrac{1}{2}(R^2-r^2)-r^2\log(R/r)].$$

PROBLEM 4. The same as Problem 3, but for a plate with supported edges.

SOLUTION.

$$\zeta = \frac{3f(1-\sigma^2)}{4\pi Eh^3}\left[\frac{3+\sigma}{1+\sigma}(R^2-r^2)-2r^2\log\frac{R}{r}\right].$$

PROBLEM 5. Determine the deflection of a circular plate suspended by its centre and in a gravitational field.

SOLUTION. The equation for ζ and its general solution are the same as in Problem 1. Since the displacement at the centre is $\zeta = 0$, we have $c = 0$. The constants a and b are determined from the boundary conditions (12.6) and (12.7), which are, for circular symmetry,

$$\frac{\mathrm{d}\triangle\zeta}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2\zeta}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right) = 0, \qquad \frac{\mathrm{d}^2\zeta}{\mathrm{d}r^2}+\frac{\sigma}{r}\frac{\mathrm{d}\zeta}{\mathrm{d}r} = 0.$$

The result is

$$\zeta = \beta r^2\left[r^2+8R^2\log\frac{R}{r}+2R^2\frac{3+\sigma}{1+\sigma}\right].$$

PROBLEM 6. A thin layer (of thickness h) is torn off a body by external forces acting against surface tension forces at the surface of separation. With given external forces, equilibrium is established for a definite area of the surface separated and a definite shape of the layer removed (Fig. 5). Derive a formula relating the surface tension to the shape of the layer removed.†

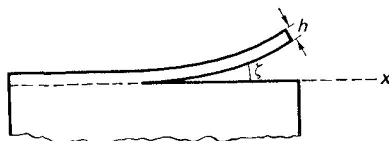


FIG. 5

SOLUTION. The layer removed can be regarded as a plate with one edge (the line of separation) clamped. The bending moment on the layer is given by formula (12.10). The work done by this moment when the length of the separated surface increases by δx is

$$M\partial\delta\zeta/\partial x = M\delta x\partial^2\zeta/\partial x^2$$

(the work of the bending force F itself is a second-order quantity). The equilibrium condition is that this work should be equal to the change in the surface energy, i.e. to $2\alpha\delta x$, where α is

† This problem was discussed by I. V. OBREIMOV (1930) in connection with a method which he developed for measuring the surface tension of mica. The measurements which he made by this method were the first direct measurements of the surface tension of solids.

the surface-tension coefficient, the factor 2 allowing for the creation of two free surfaces by the separation. Thus

$$\alpha = \frac{Eh^3}{24(1-\sigma^2)}\left(\frac{\partial^2\zeta}{\partial x^2}\right)^2.$$

§13. Longitudinal deformations of plates

Longitudinal deformations occurring in the plane of the plate, and not resulting in any bending, form a special case of deformations of thin plates. Let us derive the equations of equilibrium for such deformations.

If the plate is sufficiently thin, the deformation may be regarded as uniform over its thickness. The strain tensor is then a function of x and y only (the xy-plane being that of the plate) and is independent of z. Longitudinal deformations of a plate are usually caused either by forces applied to its edges or by body forces in its plane. The boundary conditions on both surfaces of the plate are then $\sigma_{ik}n_k = 0$, or, since the normal vector is parallel to the z-axis, $\sigma_{iz} = 0$, i.e. $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. It should be noticed, however, that in the approximate theory given below these conditions continue to hold even when the external tension forces are applied to the surfaces of the plate, since these forces are still small compared with the resulting longitudinal internal stresses (σ_{xx}, σ_{yy}, σ_{xy}) in the plate. Since they are zero at both surfaces, the quantities σ_{xz}, σ_{yz}, σ_{zz} must be small throughout the thickness of the plate, and we can therefore take them as approximately zero everywhere in the plate.

Equating to zero the expressions (11.2), we obtain the relations

$$u_{zz} = -\sigma(u_{xx}+u_{yy})/(1-\sigma), \qquad u_{xz} = u_{yz} = 0. \tag{13.1}$$

Substituting in the general formulae (5.13), we obtain for the non-zero components of the stress tensor

$$\left.\begin{aligned}\sigma_{xx} &= \frac{E}{1-\sigma^2}(u_{xx}+\sigma u_{yy}),\\ \sigma_{yy} &= \frac{E}{1-\sigma^2}(u_{yy}+\sigma u_{xx}),\\ \sigma_{xy} &= \frac{E}{1+\sigma}u_{xy}.\end{aligned}\right\} \tag{13.2}$$

It should be noticed that the formal transformation

$$E \to E/(1-\sigma^2), \qquad \sigma \to \sigma/(1-\sigma) \tag{13.3}$$

converts these expressions into those which give the relation between the stresses σ_{xx}, σ_{xy}, σ_{yy} and the strains u_{xx}, u_{yy}, u_{zz} for a plane deformation (formulae (5.13) with $u_{zz} = 0$).

Having thus eliminated the displacement u_z, we can regard the plate as a two-dimensional medium (an "elastic plane"), of zero thickness, and take

the displacement vector **u** to be a two-dimensional vector with components u_x and u_y. If P_x and P_y are the components of the external body force per unit area of the plate, the general equations of equilibrium are

$$h\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}\right)+P_x = 0,$$

$$h\left(\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}\right)+P_y = 0.$$

Substituting the expressions (13.2), we obtain the equations of equilibrium in the form

$$\begin{aligned} Eh\left\{\frac{1}{1-\sigma^2}\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{2(1+\sigma)}\frac{\partial^2 u_x}{\partial y^2}+\frac{1}{2(1-\sigma)}\frac{\partial^2 u_y}{\partial x \partial y}\right\}+P_x &= 0, \\ Eh\left\{\frac{1}{1-\sigma^2}\frac{\partial^2 u_y}{\partial y^2}+\frac{1}{2(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2}+\frac{1}{2(1-\sigma)}\frac{\partial^2 u_x}{\partial x \partial y}\right\}+P_y &= 0. \end{aligned} \tag{13.4}$$

These equations can be written in the two-dimensional vector form

$$\mathbf{grad}\ \mathrm{div}\ \mathbf{u}-\tfrac{1}{2}(1-\sigma)\ \mathbf{curl}\ \mathbf{curl}\ \mathbf{u} = -(1-\sigma^2)\mathbf{P}/Eh, \tag{13.5}$$

where all the vector operators are two-dimensional.

In particular, the equation of equilibrium in the absence of body forces is

$$\mathbf{grad}\ \mathrm{div}\ \mathbf{u}-\tfrac{1}{2}(1-\sigma)\ \mathbf{curl}\ \mathbf{curl}\ \mathbf{u} = 0. \tag{13.6}$$

It differs from the equation of equilibrium for a plane deformation of a body infinite in the z-direction (§7) only by the sign of the coefficient (in accordance with (13.3)).† As for a plane deformation, we can introduce the *stress function* defined by

$$\sigma_{xx} = \partial^2\chi/\partial y^2, \qquad \sigma_{xy} = -\partial^2\chi/\partial x \partial y, \qquad \sigma_{yy} = \partial^2\chi/\partial x^2, \tag{13.7}$$

whereby we automatically satisfy the equations of equilibrium in the form

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y} = 0.$$

The stress function, as before, satisfies the biharmonic equation, since for $\triangle\chi$ we have

$$\triangle\chi = \sigma_{xx}+\sigma_{yy} = E(u_{xx}+u_{yy})/(1-\sigma) = \{E/(1-\sigma)\}\ \mathrm{div}\ \mathbf{u};$$

this differs only by a factor from the result for a plane deformation.

It may be pointed out that the stress distribution in a plate deformed by given forces applied to its edges is independent of the elastic constants of the

† A deformation homogeneous in the z-direction for which $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$ everywhere is sometimes called a state of *plane stress*, as distinct from a plane deformation, for which $u_{zx} = u_{zy} = u_{zz} = 0$ everywhere.

material. For these constants appear neither in the biharmonic equation satisfied by the stress function, nor in the formulae (13.7) which determine the components σ_{ik} from that function (nor, therefore, in the boundary conditions at the edges of the plate).

PROBLEMS

Problem 1. Determine the deformation of a plane disc rotating uniformly about an axis through its centre perpendicular to its plane.

Solution. The required solution differs only in the constant coefficients from the solution obtained in §7, Problem 5, for the plane deformation of a rotating cylinder. The radial displacement $u_r = u(r)$ is given by the formula

$$u = \frac{\rho\Omega^2(1-\sigma^2)}{8E} r \left(\frac{3+\sigma}{1+\sigma}R^2 - r^2\right).$$

This is the expression which gives that of §7, Problem 5, if the substitution (13.3) is made.

Problem 2. Determine the deformation of a semi-infinite plate (with a straight edge) under the action of a concentrated force in its plane, applied to a point on the edge.

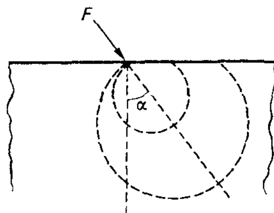


Fig. 6

Solution. We take polar co-ordinates, with the angle ϕ measured from the direction of the applied force; it takes values from $-(\frac{1}{2}\pi+\alpha)$ to $\frac{1}{2}\pi-\alpha$, where α is the angle between the direction of the force and the normal to the edge of the plate (Fig. 6). At every point of the edge except that where the force is applied (the origin) we must have $\sigma_{\phi\phi} = \sigma_{r\phi} = 0$. Using the expressions for $\sigma_{\phi\phi}$ and $\sigma_{r\phi}$ obtained in §7, Problem 11, we find that the stress function must therefore satisfy the conditions

$$\frac{\partial\chi}{\partial r} = \text{constant}, \qquad \frac{1}{r}\frac{\partial\chi}{\partial\phi} = \text{constant, for } \phi = -(\tfrac{1}{2}\pi+\alpha), (\tfrac{1}{2}\pi-\alpha).$$

Both conditions are satisfied if $\chi = rf(\phi)$. With this substitution, the biharmonic equation

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial\phi^2}\right\}^2 \chi = 0$$

gives solutions for $f(\phi)$ of the forms $\sin\phi$, $\cos\phi$, $\phi\sin\phi$, $\phi\cos\phi$. The first two of these lead to stresses which are zero identically. The solution which gives the correct value for the force applied at the origin is

$$\chi = -(F/\pi)r\phi\sin\phi, \qquad \sigma_{rr} = -(2F/\pi r)\cos\phi, \qquad \sigma_{r\phi} = \sigma_{\phi\phi} = 0, \qquad (1)$$

where F is the force per unit thickness of the plate. For, projecting the internal stresses on directions parallel and perpendicular to the force **F**, and integrating over a small semicircle

$$\int \sigma_{rr} r \sin\phi \, d\phi = 0,$$

i.e. the values required to balance the external force applied at the origin.

Formulae (1) determine the required stress distribution. It is purely radial: only a radial compression force acts on any area perpendicular to the radius. The lines of equal stress are the circles $r = d \cos \phi$, which pass through the origin and whose centres lie on the line of action of the force **F** (Fig. 6).

The components of the strain tensor are $u_{rr} = \sigma_{rr}/E$, $u_{\phi\phi} = -\sigma\sigma_{rr}/E$, $u_{r\phi} = 0$. From these we find by integration (using the expressions (1.8) for the components u_{ik} in polar co-ordinates) the displacement vector:

$$u_r = -\frac{2F}{\pi E}\log(r/a)\cos\phi - \frac{(1-\sigma)F}{\pi E}\phi\sin\phi,$$

$$u_\phi = \frac{2\sigma F}{\pi E}\sin\phi + \frac{2F}{\pi E}\log(r/a)\sin\phi + \frac{(1-\sigma)F}{\pi E}(\sin\phi - \phi\cos\phi).$$

Here the constants of integration have been chosen so as to give zero displacement (translation and rotation) of the plate as a whole: an arbitrarily chosen point at a distance a from the origin on the line of action of the force is assumed to remain fixed.

Using the solution obtained above, we can obtain the solution for any distribution of forces acting on the edge of the plate (cf. §8). It is, of course, inapplicable in the immediate neighbourhood of the origin.

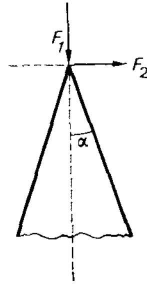


Fig. 7

where r_1, ϕ_1 and r_2, ϕ_2 are the polar co-ordinates of an arbitrary point P with origins at A and B respectively. These are the stresses due to a normal force F applied to a point on the edge of a half-plane; see Problem 2. The third distribution, $\sigma^{(3)}_{ik} = (F/\pi R)\delta_{ik}$, is a uniform extension of definite intensity. For, if the point P is on the edge of the disc, we have $r_1 = 2R\cos\phi_1$, $r_2 = 2R\cos\phi_2$, so that $\sigma^{(1)}_{r_1 r_1} = \sigma^{(2)}_{r_2 r_2} = -F/\pi R$. Since the directions of r_1 and r_2 at this point are perpendicular, we see that the first two stress distributions give a uniform compression on the edge of the disc. These forces can be just balanced by the uniform tension given by the third system, so that the edge of the disc is free from stress, as it should be.

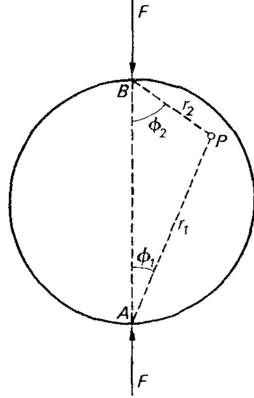


Fig. 8

Problem 5. Determine the stress distribution in an infinite sheet with a circular aperture (of radius R) under uniform tension.

Solution. The uniform tension of a continuous sheet corresponds to stresses $\sigma^{(0)}_{xx} = T$, $\sigma^{(0)}_{yy} = \sigma^{(0)}_{xy} = 0$, where T is the tension force. These in turn correspond to the stress function $\chi^{(0)} = \frac{1}{2}Ty^2 = \frac{1}{2}Tr^2\sin^2\phi = \frac{1}{4}Tr^2(1-\cos 2\phi)$. When there is a circular aperture (with the centre as the origin of polar co-ordinates r, ϕ), we seek the stress function in the form $\chi = \chi^{(0)} + \chi^{(1)}$, $\chi^{(1)} = f(r) + F(r)\cos 2\phi$. The integral of the biharmonic equation which is independent of ϕ is of the form $f(r) = ar^2\log r + br^2 + c\log r$, and in the integral proportional to $\cos 2\phi$ we have $F(r) = dr^2 + er^4 + g/r^2$. The constants are determined by the conditions $\sigma^{(1)}_{ik} = 0$ for $r = \infty$ and $\sigma_{rr} = \sigma_{r\phi} = 0$ for $r = R$. The result is

$$\chi^{(1)} = \tfrac{1}{2}TR^2\left\{-\log r + \left(1 - \frac{R^2}{2r^2}\right)\cos 2\phi\right\},$$

§14. Large deflections of plates

The theory of the bending of thin plates given in §§11–13 is applicable only to fairly small deflections. Anticipating the result given below, it may be mentioned here that the condition for that theory to be applicable is that the deflection ζ is small compared with the thickness h of the plate. Let us now derive the equations of equilibrium for a plate undergoing large deflections. The deflection ζ is not now supposed small compared with h. It should be emphasised, however, that the deformation itself must still be small, in the sense that the components of the strain tensor must be small. In practice, this usually implies the condition $\zeta \ll l$, i.e. the deflection must be small compared with the dimension l of the plate.

The bending of a plate in general involves a stretching of it.† For small deflections this stretching can be neglected. For large deflections, however, this is not possible; there is therefore no neutral surface in a plate undergoing large deflections. The existence of a stretching which accompanies the bending is peculiar to plates, and distinguishes them from thin rods, which can undergo large deflections without any general stretching. This property of plates is a purely geometrical one. For example, let a flat circular plate be bent into a segment of a spherical surface. If the bending is such that the circumference of the plate remains constant, its diameter must increase. If the diameter is constant, on the other hand, the circumference must be reduced.

The energy (11.6), which may be called the *pure bending energy*, is only the part of the total energy which arises from the non-uniformity of the tension and compression through the thickness of the plate, in the absence of any general stretching. The total energy includes also a part due to this general stretching; this may be called the *stretching energy*.

Deformations consisting of pure bending and pure stretching have been considered in §§11–13. We can therefore use the results obtained in these sections. It is not necessary to consider the structure of the plate across its thickness, and we can regard it as a two-dimensional surface of negligible thickness.

We first derive an expression for the strain tensor pertaining to the stretching of a plate (regarded as a surface) which is simultaneously bent and stretched in its plane. Let $\mathbf{u}$ be the two-dimensional displacement vector (with components u_x, u_y) for pure stretching; ζ, as before, denotes the transverse displacement in bending. Then the element of length $dl = \sqrt{(dx^2+dy^2)}$ of the undeformed plate is transformed by the deformation into an element dl', whose square is given by $dl'^2 = (dx+du_x)^2+(dy+du_y)^2+d\zeta^2$. Putting here $du_x = (\partial u_x/\partial x)\,dx+(\partial u_x/\partial y)\,dy$, and similarly for du_y and $d\zeta$, we obtain to within higher-order terms $dl'^2 = dl^2+2u_{\alpha\beta}dx_\alpha dx_\beta$, where the two-dimensional strain tensor is defined as

$$u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right)+\frac{1}{2}\frac{\partial \zeta}{\partial x_\alpha}\frac{\partial \zeta}{\partial x_\beta}. \tag{14.1}$$

† An exception is, for instance, the bending of a flat plate into a cylindrical surface.

(In this and the following sections, Greek suffixes take the two values x and y; as usual, summation over repeated suffixes is understood.) The terms quadratic in the derivatives of u_α are here omitted; the same cannot, of course, be done with the derivatives of ζ, since there are no corresponding first-order terms.

The stress tensor $\sigma_{\alpha\beta}$ due to the stretching of the plate is given by formula (13.2), in which $u_{\alpha\beta}$ must be replaced by the total strain tensor given by formula (14.1). The pure bending energy is given by formula (11.6), and can be written $\int \Psi_1(\zeta)\, dx\, dy$, where $\Psi_1(\zeta)$ denotes the integrand in (11.6). The stretching energy per unit volume of the plate is, by the general formulae, $\frac{1}{2}u_{\alpha\beta}\sigma_{\alpha\beta}$. The energy per unit surface area is obtained by multiplying by h, so that the total stretching energy can be written $\int \Psi_2(u_{\alpha\beta})\, df$, where

$$\Psi_2 = \tfrac{1}{2}hu_{\alpha\beta}\sigma_{\alpha\beta}. \tag{14.2}$$

Thus the total free energy of a plate undergoing large deflections is

$$F_{\text{pl}} = \int\{\Psi_1(\zeta)+\Psi_2(u_{\alpha\beta})\}\, df. \tag{14.3}$$

Before deriving the equations of equilibrium, let us estimate the relative magnitude of the two parts of the energy. The first derivatives of ζ are of the order of ζ/l, where l is the dimension of the plate, and the second derivatives are of the order of ζ/l^2. Hence we see from (11.6) that $\Psi_1 \sim Eh^3\zeta^2/l^4$. The order of magnitude of the tensor components $u_{\alpha\beta}$ is ζ^2/l^2, and so $\Psi_2 \sim Eh\zeta^4/l^4$. A comparison shows that the neglect of Ψ_2 in the approximate theory of the bending of plates is valid only if $\zeta^2 \ll h^2$.

The condition of minimum energy is $\delta F+\delta U = 0$, where U is the potential energy in the field of the external forces. We shall suppose that the external stretching forces, if any, can be neglected in comparison with the bending forces. (This is always valid unless the stretching forces are very large, since a thin plate is much more easily bent than stretched.) Then we have for δU the same expression as in §12: $\delta U = -\int P\delta\zeta\, df$, where P is the external force per unit area of the plate. The variation of the integral $\int \Psi_1\, df$ has already been calculated in §12, and is

$$\delta\int\Psi_1\, df = \frac{Eh^3}{12(1-\sigma^2)}\int \triangle^2\zeta\delta\zeta\, df.$$

The contour integrals in (12.3) are omitted, since they give only the boundary conditions on the equation of equilibrium, and not that equation itself, which is of interest here.

Finally, let us calculate the variation of the integral $\int \Psi_2\, df$. The variation must be taken both with respect to the components of the vector $\mathbf{u}$ and with respect to ζ. We have

$$\delta\int\Psi_2\, df = \int\frac{\partial\Psi_2}{\partial u_{\alpha\beta}}\delta u_{\alpha\beta}\, df.$$

The derivatives of the free energy per unit volume with respect to $u_{\alpha\beta}$ are

$\sigma_{\alpha\beta}$; hence $\partial\Psi_2/\partial u_{\alpha\beta} = h\sigma_{\alpha\beta}$. Substituting also for $u_{\alpha\beta}$ the expression (14.1), we obtain

$$\delta\int\Psi_2\,df = h\int\sigma_{\alpha\beta}\delta u_{\alpha\beta}\,df$$

$$= \tfrac{1}{2}h\int\sigma_{\alpha\beta}\left\{\frac{\partial\delta u_\alpha}{\partial x_\beta}+\frac{\partial\delta u_\beta}{\partial x_\alpha}+\frac{\partial\zeta}{\partial x_\alpha}\frac{\partial\delta\zeta}{\partial x_\beta}+\frac{\partial\delta\zeta}{\partial x_\alpha}\frac{\partial\zeta}{\partial x_\beta}\right\}df,$$

or, by the symmetry of $\sigma_{\alpha\beta}$,

$$\delta\int\Psi_2\,df = h\int\sigma_{\alpha\beta}\left\{\frac{\partial\delta u_\alpha}{\partial x_\beta}+\frac{\partial\delta\zeta}{\partial x_\beta}\frac{\partial\zeta}{\partial x_\alpha}\right\}df.$$

Integrating by parts, we obtain

$$\delta\int\Psi_2\,df = -h\int\left\{\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\delta u_\alpha+\frac{\partial}{\partial x_\beta}\left(\sigma_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\right)\delta\zeta\right\}df.$$

The contour integrals along the circumference of the plate are again omitted.

Collecting the above results, we have

$$\delta F_{\rm pl}+\delta U = \int\left[\left\{\frac{Eh^3}{12(1-\sigma^2)}\triangle^2\zeta-h\frac{\partial}{\partial x_\beta}\left(\sigma_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\right)-P\right\}\delta\zeta-h\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\delta u_\alpha\right]df = 0.$$

In order that this relation should be satisfied identically, the coefficients of $\delta\zeta$ and δu_α must each be zero. Thus we obtain the equations

$$\frac{Eh^3}{12(1-\sigma^2)}\triangle^2\zeta-h\frac{\partial}{\partial x_\beta}\left(\sigma_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\right) = P, \tag{14.4}$$

$$\partial\sigma_{\alpha\beta}/\partial x_\beta = 0. \tag{14.5}$$

The unknown functions here are the two components u_x, u_y of the vector $\mathbf{u}$ and the transverse displacement ζ. The solution of the equations gives both the form of the bent plate (i.e. the function $\zeta(x, y)$) and the extension resulting from the bending. Equations (14.4) and (14.5) can be somewhat simplified by introducing the function χ related to $\sigma_{\alpha\beta}$ by (13.7). Equation (14.4) then becomes

$$\frac{Eh^3}{12(1-\sigma^2)}\triangle^2\zeta-h\left(\frac{\partial^2\chi}{\partial y^2}\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\chi}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}-2\frac{\partial^2\chi}{\partial x\partial y}\frac{\partial^2\zeta}{\partial x\partial y}\right) = P. \tag{14.6}$$

Equations (14.5) are satisfied automatically by the expressions (13.7). Hence another equation is needed; this can be obtained by eliminating u_α from the relations (13.7) and (13.2).

To do this, we proceed as follows. We express $u_{\alpha\beta}$ in terms of $\sigma_{\alpha\beta}$, obtaining from (13.2)

$$u_{xx} = (\sigma_{xx}-\sigma\sigma_{yy})/E, \qquad u_{yy} = (\sigma_{yy}-\sigma\sigma_{xx})/E, \qquad u_{xy} = (1+\sigma)\sigma_{xy}/E.$$

Substituting here the expression (14.1) for $u_{\alpha\beta}$, and (13.7) for $\sigma_{\alpha\beta}$, we find the equations

$$\frac{\partial u_x}{\partial x}+\frac{1}{2}\left(\frac{\partial \zeta}{\partial x}\right)^2 = \frac{1}{E}\left(\frac{\partial^2 \chi}{\partial y^2} - \sigma\frac{\partial^2 \chi}{\partial x^2}\right),$$

$$\frac{\partial u_y}{\partial y}+\frac{1}{2}\left(\frac{\partial \zeta}{\partial y}\right)^2 = \frac{1}{E}\left(\frac{\partial^2 \chi}{\partial x^2} - \sigma\frac{\partial^2 \chi}{\partial y^2}\right),$$

$$\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}+\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{dy} = -\frac{2(1+\sigma)}{E}\frac{\partial^2 \chi}{\partial x \partial y}.$$

We take $\partial^2/\partial y^2$ of the first, $\partial^2/\partial x^2$ of the second, $-\partial^2/\partial x\partial y$ of the third, and add. The terms in u_x and u_y then cancel, and we have the equation

$$\triangle^2\chi+E\left\{\frac{\partial^2 \zeta}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2} - \left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2\right\} = 0. \tag{14.7}$$

Equations (14.6) and (14.7) form a complete system of equations for large deflections of thin plates (A. FÖPPL 1907). These equations are very complicated, and cannot be solved exactly, even in very simple cases. It should be noticed that they are non-linear.

We may mention briefly a particular case of deformations of thin plates, that of membranes. A *membrane* is a thin plate subject to large external stretching forces applied at its circumference. In this case we can neglect the additional longitudinal stresses caused by bending of the plate, and therefore suppose that the components of the tensor $\sigma_{\alpha\beta}$ are simply equal to the constant external stretching forces. In equation (14.4) we can then neglect the first term in comparison with the second, and we obtain the equation of equilibrium

$$h\sigma_{\alpha\beta}\frac{\partial^2 \zeta}{\partial x_\alpha \partial x_\beta}+P = 0, \tag{14.8}$$

with the boundary condition that $\zeta = 0$ at the edge of the membrane. This is a linear equation. The case of isotropic stretching, when the extension of the membrane is the same in all directions, is particularly simple. Let T be the absolute magnitude of the stretching force per unit length of the edge of the membrane. Then $h\sigma_{\alpha\beta} = T\delta_{\alpha\beta}$, and we obtain the equation of equilibrium in the form

$$T\triangle\zeta+P = 0. \tag{14.9}$$

PROBLEMS

PROBLEM 1. Determine the deflection of a plate as a function of the force on it when $\zeta \gg h$.

SOLUTION. An estimate of the terms in equation (14.7) shows that $\chi \sim E\zeta^2$. For $\zeta \gg h$, the first term in (14.6) is small compared with the second, which is of the order of magnitude $h\zeta\chi/l^4 \sim Eh\zeta^3/l^4$ (l being the dimension of the plate). If this is comparable with the external

force P, we have $\zeta \sim (l^4P/Eh)^{\frac{1}{3}}$. Hence, in particular, we see that ζ is proportional to the cube root of the force.

PROBLEM 2. Determine the deformation of a circular membrane (of radius R) placed horizontally in a gravitational field.

SOLUTION. We have $P = \rho gh$; in polar co-ordinates, (14.9) becomes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right) = -\frac{\rho gh}{T}.$$

The solution finite for $r = 0$ and zero for $r = R$ is $\zeta = \rho gh(R^2 - r^2)/4T$.

§15. Deformations of shells

In discussing hitherto the deformations of thin plates, we have always assumed that the plate is flat in its undeformed state. However, deformations of plates which are curved in the undeformed state (called *shells*) have properties which are fundamentally different from those of the deformations of flat plates.

The stretching which accompanies the bending of a flat plate is a second-order effect in comparison with the bending deflection itself. This is seen, for example, from the fact that the strain tensor (14.1), which gives this stretching, is quadratic in ζ. The situation is entirely different in the deformation of shells: here the stretching is a first-order effect, and therefore is important even for small bending deflections. This property is most easily seen from a simple example, that of the uniform stretching of a spherical shell. If every point undergoes the same radial displacement ζ, the length of the equator increases by $2\pi\zeta$. The relative extension is $2\pi\zeta/2\pi R = \zeta/R$, and hence the strain tensor also is proportional to the first power of ζ. This effect tends to zero as $R \to \infty$, i.e. as the curvature tends to zero, and is therefore due to the curvature of the shell.

Let R be the order of magnitude of the radius of curvature of the shell, which is usually of the same order as its dimension. Then the strain tensor for the stretching which accompanies the bending is of the order of ζ/R, the corresponding stress tensor is $\sim E\zeta/R$, and the deformation energy per unit area is, by (14.2), of the order of $Eh(\zeta/R)^2$. The pure bending energy, on the other hand, is of the order of $Eh^3\zeta^2/R^4$, as before. We see that the ratio of the two is of the order of $(R/h)^2$, i.e. it is very large. It should be emphasised that this is true whatever the ratio of the bending deflection ζ to the thickness h, whereas in the bending of flat plates the stretching was important only for $\zeta \gtrsim h$.

In some cases there may be a special type of bending of the shell in which no stretching occurs. For example, a cylindrical shell (open at both ends) can be deformed without stretching if all the generators remain parallel (i.e. if the shell is, as it were, compressed along some generator). Such deformations without stretching are geometrically possible if the shell has free edges (i.e. is not closed) or if it is closed but its curvature has opposite

signs at different points. For example, a closed spherical shell cannot be bent without being stretched, but if a hole is cut in it (the edge of the hole not being fixed), then such a deformation becomes possible. Since the pure bending energy is small compared with the stretching energy, it is clear that, if any given shell permits deformation without stretching, then such deformations will, in general, actually occur when arbitrary external forces act on the shell. The requirement that the bending is unaccompanied by stretching places considerable restrictions on the possible displacements u_α. These restrictions are purely geometrical, and can be expressed as differential equations, which must be contained in the complete system of equilibrium equations for such deformations. We shall not pause to discuss this question further.

If, however, the deformation of the shell involves stretching, then the tensile stresses are in general large compared with the bending stresses, which may be neglected. Shells for which this is done are called *membranes*.

The stretching energy of a shell can be calculated as the integral

$$F_{\mathrm{pl}} = \tfrac{1}{2}h \int u_{\alpha\beta}\sigma_{\alpha\beta}\,\mathrm{d}f, \tag{15.1}$$

taken over the surface. Here $u_{\alpha\beta}$ (α, $\beta = 1, 2$) is the two-dimensional strain tensor in the appropriate curvilinear co-ordinates, and the stress tensor $\sigma_{\alpha\beta}$ is related to $u_{\alpha\beta}$ by formulae (13.2), which can be written, in two-dimensional tensor notation, as

$$\sigma_{\alpha\beta} = E[(1-\sigma)u_{\alpha\beta} + \sigma\delta_{\alpha\beta}u_{\gamma\gamma}]/(1-\sigma^2). \tag{15.2}$$

A case requiring special consideration is that where the shell is subjected to the action of forces applied to points or lines on the surface and directed through the shell. These may be, in particular, the reaction forces on the shell at points (or lines) where it is fixed. The concentrated forces result in a bending of the shell in small regions near the points where they are applied; let d be the dimension of such a region for a force f applied at a point (so that its area is of the order of d^2). Since the deflection ζ varies considerably over a distance d, the bending energy per unit area is of the order of $Eh^3\zeta^2/d^4$, and the total bending energy (over an area $\sim d^2$) is of the order of $Eh^3\zeta^2/d^2$. The strain tensor for the stretching is again $\sim \zeta/R$, and the total stretching energy due to the concentrated forces is $\sim Eh\zeta^2d^2/R^2$. Since the bending energy increases and the stretching energy decreases with decreasing d, it is clear that both energies must be taken into account in determining the deformation near the point of application of the forces. The size d of the region of bending is given in order of magnitude by the condition that the sum of these energies is a minimum, whence

$$d \sim \sqrt{(hR)}. \tag{15.3}$$

The energy $\sim Eh^2\zeta^2/R$. Varying this with respect to ζ and equating the result to the work done by the force f, we find the deflection $\zeta \sim fR/Eh^2$.

However, if the forces acting on the shell are sufficiently large, the shape of the shell may be considerably changed by bulges which form in it. The determination of the deformation as a function of the applied loads requires special investigation in this unusual case.†

Let a convex shell (with edges fixed in such a way that it is geometrically rigid) be subjected to the action of a large concentrated force f along the inward normal. For simplicity we shall assume that the shell is part of a sphere of radius R. The region of the bulge will be a spherical cap which is almost a mirror image of its original shape (Fig. 9 shows a meridional section of the shell). The problem is to determine the size of the bulge as a function of the force.

The major part of the elastic energy is concentrated in a narrow strip near the edge of the bulge, where the bending of the shell is relatively large; we shall call this the *bending strip* and denote its width by d. This energy may be estimated, assuming that the radius r of the bulge region is much less than R, so that the angle $\alpha \ll 1$ (Fig. 9). Then $r = R\sin\alpha \sim R\alpha$, and the depth of the

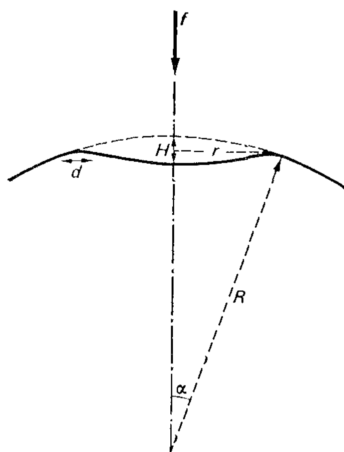


FIG. 9

bulge $H = 2R(1-\cos\alpha) \sim R\alpha^2$. Let ζ denote the displacement of points on the shell in the bending strip. Just as previously, we find that the energies of bending along the meridian and of stretching along the circle of latitude‡ per unit surface area are respectively, in order of magnitude, $Eh^3\zeta^2/d^4$ and

† The results given below are due to A. V. POGORELOV (1960). A more precise analysis of the problem together with some similar ones is given in his book *Teoriya obolochek pri zakriticheskikh deformatsiyakh* (*Theory of Shells at Supercritical Deformations*), Moscow 1965.

‡ The curvature of the shell does not affect the bending along the meridian in the first approximation, so that this bending occurs without any general stretching along the meridian, as in the cylindrical bending of a flat plate.

$Eh\zeta^2/R^2$. The order of magnitude of the displacement ζ is in this case determined geometrically: the direction of the meridian changes by an angle $\sim\alpha$ over the width d, and so $\zeta\sim\alpha d\sim rd/R$. Multiplying by the area of the bending strip ($\sim rd$), we obtain the energies Eh^3r^3/R^2d and Ehd^3r^3/R^4. The condition for their sum to be a minimum again gives $d\sim\sqrt{(hR)}$, and the total elastic energy is then $\sim Er^3(h/R)^{5/2}$, or†

$$\text{constant}\times E h^{5/2}.\ H^{3/2}/R. \tag{15.4}$$

In this derivation it has been assumed that $d\ll r$; formula (15.4) is therefore valid if the condition‡

$$Rh/r^2\ll 1 \tag{15.5}$$

holds.

The required relation between the depth of the bulge H and the applied force f is obtained by equating f to the derivative of the energy (15.4) with respect to H. Thus we find

$$H\sim f^2R^2/E^2h^5. \tag{15.6}$$

It should be noticed that this relation is non-linear.

Finally, let the deformation (bulge) of the shell occur under a uniform external pressure p. In this case the work done is $p\Delta V$, where $\Delta V\sim Hr^2\sim H^2R$ is the change in the volume within the shell when the bulge occurs. Equating to zero the derivative with respect to H of the total free energy (the difference between the elastic energy (15.4) and this work), we obtain

$$H\sim h^5E^2/R^4p^2. \tag{15.7}$$

The inverse variation (H increasing when p decreases) shows that in this case the bulge is unstable. The value of H given by formula (15.7) corresponds to unstable equilibrium for a given p: bulges with larger values of H grow of their own accord, while smaller ones shrink (it is easy to verify that (15.7) corresponds to a maximum and not a minimum of the total free energy). There is a critical value p_{cr} of the external load beyond which even small changes in the shape of the shell increases in size spontaneously. This value may be defined as that which gives $H\sim h$ in (15.7):

$$p_{cr}\sim Eh^2/R^2. \tag{15.8}$$

We shall add to the above brief account of shell theory only a few simple examples in the following Problems.

---

† A more accurate calculation shows that the constant coefficient is $1.2\,(1-\sigma^2)^{-3/4}$.

‡ When a bulge is formed, the outer layers of a spherical segment become the inner ones and are therefore compressed, while the inner layers become the outer ones and are stretched. The relative extension (or compression) $\sim h/R$, and so the corresponding total energy in the region of the bulge $\sim E(h/R)^2hr^2$. With the condition (15.5) it is in fact small in comparison with the energy (15.4) in the bending strip.

PROBLEMS

Problem 1. Derive the equations of equilibrium for a spherical shell (of radius R) deformed symmetrically about an axis through its centre.

Solution. We take as two-dimensional co-ordinates on the surface of the shell the angles θ, ϕ in a system of spherical polar co-ordinates, whose origin is at the centre of the sphere and polar axis along the axis of symmetry of the deformed shell.

Let P_r be the external radial force per unit surface area. This force must be balanced by a radial resultant of internal stresses acting tangentially on an element of the shell. The condition is

$$h(\sigma_{\phi\phi}+\sigma_{\theta\theta})/R = P_r. \tag{1}$$

This equation is exactly analogous to Laplace's equation for the pressure difference between two media caused by surface tension at the surface of separation.

Next, let $Q_z(\theta)$ be the resultant of all external forces on the part of the shell lying above the co-latitude θ; this resultant is along the polar axis. The force $Q_z(\theta)$ must be balanced by the projection on the polar axis of the stresses $2\pi Rh\sigma_{\theta\theta}\sin\theta$ acting on the cross-section $2\pi Rh\sin\theta$ of the shell at that latitude. Hence

$$2\pi Rh\sigma_{\theta\theta}\sin^2\theta = Q_z(\theta). \tag{2}$$

Equations (1) and (2) determine the stress distribution, and the strain tensor is then given by the formulae

$$u_{\theta\theta} = (\sigma_{\theta\theta}-\sigma\sigma_{\phi\phi})/E, \qquad u_{\phi\phi} = (\sigma_{\phi\phi}-\sigma\sigma_{\theta\theta})/E, \qquad u_{\theta\phi} = 0. \tag{3}$$

Finally, the displacement vector is obtained from the equations

$$u_{\theta\theta} = \frac{1}{R}\left(\frac{\mathrm{d}u_\theta}{\mathrm{d}\theta}+u_r\right), \qquad u_{\theta\phi} = \frac{1}{R}(u_\theta\cot\theta+u_r). \tag{4}$$

Problem 2. Determine the deformation under its own weight of a hemispherical shell convex upwards, the edge of which moves freely on a horizontal support (Fig. 10).

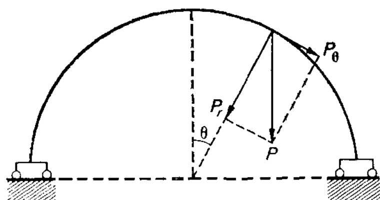


Fig. 10

Solution. We have $P_r = -\rho gh\cos\theta$, $Q_z = -2\pi R^2\rho gh(1-\cos\theta)$; Q_z is the total weight of the shell above the circle of co-latitude θ. From (1) and (2) of Problem 1 we find

$$\sigma_{\theta\theta} = -\frac{R\rho g}{1+\cos\theta}, \qquad \sigma_{rr} = R\rho g\left(\frac{1}{1+\cos\theta}-\cos\theta\right).$$

From (3) we calculate $u_{\phi\phi}$ and $u_{\theta\theta}$, and then obtain u_θ and u_r from (4); the constant in the integration of the first equation (4) is chosen so that for $\theta = \frac{1}{2}\pi$ we have $u_\theta = 0$. The result is

$$u_\theta = \frac{R^2\rho g(1+\sigma)}{E}\left\{\frac{\cos\theta}{1+\cos\theta}+\log(1+\cos\theta)\right\}\sin\theta,$$

$$u_r = \frac{R^2\rho g(1+\sigma)}{E}\left\{1-\frac{2+\sigma}{1+\sigma}\cos\theta-\cos\theta\log(1+\cos\theta)\right\}.$$

The value of u_r for $\theta = \frac{1}{2}\pi$ gives the horizontal displacement of the support.

PROBLEM 3. Determine the deformation of a hemispherical shell with clamped edges, convex downwards and filled with liquid (Fig. 11); the weight of the shell itself can be neglected in comparison with that of the liquid.

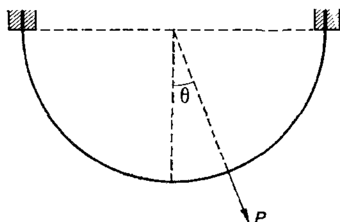


FIG. 11

SOLUTION. We have

$$P_r = \rho_0 g R \cos\theta, \qquad P_\theta = 0,$$

$$Q_z = 2\pi R^2 \int_0^\theta P_r \cos\theta \sin\theta \, d\theta = \frac{2}{3}\pi R^3 \rho_0 g (1-\cos^3\theta),$$

where ρ_0 is the density of the liquid. We find from (1) and (2) of Problem 1

$$\sigma_{\theta\theta} = \frac{R^2\rho_0 g}{3h}\cdot\frac{1-\cos^3\theta}{\sin^2\theta}, \qquad \sigma_{\phi\phi} = \frac{R^2\rho_0 g}{3h}\cdot\frac{-1+3\cos\theta-2\cos^3\theta}{\sin^2\theta}.$$

The displacements are

$$u_\theta = -\frac{R^3\rho_0 g(1+\sigma)}{3Eh}\sin\theta\left[\frac{\cos\theta}{1+\cos\theta}+\log(1+\cos\theta)\right],$$

$$u_r = \frac{R^3\rho_0 g(1+\sigma)}{3Eh}\left[\cos\theta\log(1+\cos\theta)-1+\frac{3\cos\theta}{1+\sigma}\right].$$

For $\theta = \frac{1}{2}\pi$, u_r is not zero as it should be. This means that the shell is actually so severely bent near the clamped edge that the above solution is invalid.

PROBLEM 4. A shell in the form of a spherical cap rests on a fixed support (Fig. 12). Determine the bending resulting from the weight Q of the shell.

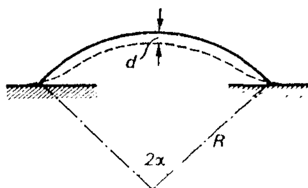


FIG. 12

SOLUTION. The main deformation occurs near the edge, which is bent as shown by the dashed line in Fig. 12. The displacement u_θ is small compared with the radial displacement $u_r \equiv \zeta$. Since ζ decreases rapidly as we move away from the supported edge, the deformation can be regarded as that of a long flat plate (of length $2\pi R \sin\alpha$). This deformation is composed of a bending and a stretching of the plate. The relative extension at each point is ζ/R (R being the radius of the shell), and therefore the stretching energy is $E\zeta^2/2R^2$ per unit volume. Using as the independent variable the distance x from the line of support, we have for the total stretching energy

$$F_{1,\mathrm{pl}} = 2\pi R \sin\alpha \frac{Eh}{2R^2}\int \zeta^2 \,\mathrm{d}x.$$

The bending energy is

$$F_{2,\mathrm{pl}} = 2\pi R \sin\alpha \frac{Eh^3}{24(1-\sigma^2)}\int \left(\frac{\mathrm{d}^2\zeta}{\mathrm{d}x^2}\right)^2 \mathrm{d}x.$$

Varying the sum $F_{\mathrm{pl}} = F_{1,\mathrm{pl}} + F_{2,\mathrm{pl}}$ with respect to ζ, we obtain

$$\frac{\mathrm{d}^4\zeta}{\mathrm{d}x^4} + \frac{12(1-\sigma^2)}{h^2R^2}\zeta = 0.$$

For $x \to \infty$, ζ must tend to zero, and for $x = 0$ we must have the boundary conditions of zero moment of the forces ($\zeta'' = 0$) and equality of the normal force and the corresponding component of the force of gravity:

$$2\pi R \sin\alpha \frac{Eh^3}{12(1-\sigma^2)}\zeta''' = Q\cos\alpha.$$

The solution which satisfies these conditions is $\zeta = Ae^{-\kappa x}\cos\kappa x$, where

$$\kappa = \left[\frac{3(1-\sigma^2)}{h^2R^2}\right]^{1/4}, \quad A = \frac{Q\cot\alpha}{Eh}\left[\frac{3R^2(1-\sigma^2)}{8\pi h^2}\right]^{1/4}.$$

The bending of the shell is

$$d = \zeta(0)\cos\alpha = A\cos\alpha.$$

§16. Torsion of rods

Let us now consider the deformation of thin rods. This differs from all the cases hitherto considered, in that the displacement vector $\mathbf{u}$ may be large even for small strains, i.e. when the tensor u_{ik} is small.† For example, when a long thin rod is slightly bent, its ends may move a considerable distance, even though the relative displacements of neighbouring points in the rod are small.

There are two types of deformation of a rod which may be accompanied by a large displacement of certain parts of it. One of these consists in bending

† The only exception is a simple extension of a rod without change of shape, in which case the vector $\mathbf{u}$ is always small if the tensor u_{ik} is small, i.e. if the extension is small.

the rod, and the other in twisting it. We shall begin by considering the latter case.

A *torsional deformation* is one in which, although the rod remains straight, each transverse section is rotated through some angle relative to those below it. If the rod is long, even a slight torsion causes sufficiently distant cross-sections to turn through large angles. The generators on the sides of the rod, which are parallel to its axis, become helical in form under torsion.

Let us consider a thin straight rod of arbitrary cross-section. We take a co-ordinate system with the z-axis along the axis of the rod and the origin somewhere inside the rod. We use also the *torsion angle* τ, which is the angle of rotation per unit length of the rod. This means that two neighbouring cross-sections at a distance dz will rotate through a relative angle $d\phi = \tau\, dz$ (so that $\tau = d\phi/dz$). The torsional deformation itself, i.e. the relative displacement of adjoining parts of the rod, is assumed small. The condition for this to be so is that the relative angle turned through by cross-sections of the rod at a distance apart of the order of its transverse dimension R is small, i.e.

$$\tau R \ll 1. \tag{16.1}$$

Let us examine a small portion of the length of the rod near the origin, and determine the displacements $\mathbf{u}$ of the points of the rod in that portion. As the undisplaced cross-section we take that given by the xy-plane. When a radius vector $\mathbf{r}$ turns through a small angle $\delta\phi$, the displacement of its end is given by

$$\delta\mathbf{r} = \delta\boldsymbol{\phi} \times \mathbf{r}, \tag{16.2}$$

where $\delta\boldsymbol{\phi}$ is a vector whose magnitude is the angle of rotation and whose direction is that of the axis of rotation. In the present case, the rotation is about the z-axis, and for points of co-ordinate z the angle of rotation relative to the xy-plane is τz (since τ can be regarded as a constant in some region near the origin). Then formula (16.2) gives for the components u_x, u_y of the displacement vector

$$u_x = -\tau z y, \qquad u_y = \tau z x. \tag{16.3}$$

When the rod is twisted, the points in it in general undergo a displacement along the z-axis also. Since for $\tau = 0$ this displacement is zero, it may be supposed proportional to τ when τ is small. Thus

$$u_z = \tau\psi(x, y), \tag{16.4}$$

where $\psi(x, y)$ is some function of x and y, called the *torsion function*. As a result of the deformation described by formulae (16.3) and (16.4), each cross-section of the rod rotates about the z-axis, and also becomes curved instead of plane. It should be noted that, by taking the origin at a particular point in the xy-plane, we "fix" a certain point in the cross-section of the rod in such a

way that it cannot move in that plane (but it can move in the z-direction). A different choice of origin would not, of course, affect the torsional deformation itself, but would give only an unimportant displacement of the rod as a whole.

Knowing $\mathbf{u}$, we can find the components of the strain tensor. Since $\mathbf{u}$ is small in the region under consideration, we can use the formula

$$u_{ik} = \tfrac{1}{2}(\partial u_i/\partial x_k + \partial u_k/\partial x_i).$$

The result is

$$u_{xx} = u_{yy} = u_{xy} = u_{zz} = 0,$$

$$u_{xz} = \tfrac{1}{2}\tau\left(\frac{\partial\psi}{\partial x} - y\right), \qquad u_{yz} = \tfrac{1}{2}\tau\left(\frac{\partial\psi}{\partial y} + x\right). \tag{16.5}$$

It should be noticed that $u_{ll} = 0$; in other words, torsion does not result in a change in volume, i.e. it is a pure shear deformation.

For the components of the stress tensor we find

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0,$$

$$\sigma_{xz} = 2\mu u_{xz} = \mu\tau\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \sigma_{yz} = 2\mu u_{yz} = \mu\tau\left(\frac{\partial\psi}{\partial y} + x\right). \tag{16.6}$$

Here it is more convenient to use the modulus of rigidity μ in place of E and σ. Since only σ_{xz} and σ_{yz} are different from zero, the general equations of equilibrium $\partial\sigma_{ik}/\partial x_k = 0$ reduce to

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} = 0. \tag{16.7}$$

Substituting (16.6), we find that the torsion function must satisfy the equation

$$\triangle\psi = 0, \tag{16.8}$$

where $\triangle$ is the two-dimensional Laplacian.

It is rather more convenient, however, to use a different auxiliary function $\chi(x, y)$, defined by

$$\sigma_{xz} = 2\mu\tau\partial\chi/\partial y, \qquad \sigma_{yz} = -2\mu\tau\partial\chi/\partial x; \tag{16.9}$$

this function satisfies more convenient boundary conditions on the circumference of the rod (see below). Comparing (16.9) and (16.6), we obtain

$$\frac{\partial\psi}{\partial x} = y + 2\frac{\partial\chi}{\partial y}, \qquad \frac{\partial\psi}{\partial y} = -x - 2\frac{\partial\chi}{\partial x}. \tag{16.10}$$

Differentiating the first of these with respect to y, the second with respect to

x, and subtracting, we obtain for the function χ the equation

$$\triangle\chi = -1. \tag{16.11}$$

To determine the boundary conditions on the surface of the rod, we note that, since the rod is thin, the external forces on its sides must be small compared with the internal stresses in the rod, and can therefore be put equal to zero in seeking the boundary conditions. This fact is exactly analogous to what we found in discussing the bending of thin plates. Thus we must have $\sigma_{ik}n_k = 0$ on the sides of the rod; since the z-direction is along the axis, $n_z = 0$, and this equation becomes

$$\sigma_{zx}n_x + \sigma_{zy}n_y = 0.$$

Substituting (16.9), we obtain

$$\frac{\partial\chi}{\partial y}n_x - \frac{\partial\chi}{\partial x}n_y = 0.$$

The components of the vector normal to a plane contour (the circumference of the rod) are $n_x = -\mathrm{d}y/\mathrm{d}l$, $n_y = \mathrm{d}x/\mathrm{d}l$, where x and y are co-ordinates of points on the contour and $\mathrm{d}l$ is an element of arc. Thus we have

$$\frac{\partial\chi}{\partial x}\mathrm{d}x + \frac{\partial\chi}{\partial y}\mathrm{d}y = \mathrm{d}\chi = 0,$$

whence χ = constant, i.e. χ is constant on the circumference. Since only the derivatives of the function χ appear in the definitions (16.9), it is clear that any constant may be added to χ. If the cross-section is singly connected, we can therefore use, without loss of generality, the boundary condition

$$\chi = 0 \tag{16.12}$$

on equation (16.11).†

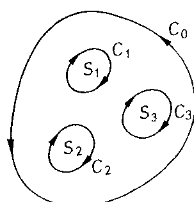


FIG. 13

For a multiply connected cross-section, however, χ will have different constant values on each of the closed curves bounding the cross-section.

† The problem of determining the torsion deformation from equation (16.11) with the boundary condition (16.12) is formally identical with that of determining the bending of a uniformly loaded plane membrane from equation (14.9).

It is useful to note also an analogy with fluid mechanics: an equation of the form (16.11) determines the velocity distribution $v(x, y)$ for a viscous fluid in a pipe, and the boundary condition (16.12) corresponds to the condition $v = 0$ at the fixed walls of the pipe (see *Fluid Mechanics*, §17).

Hence we can put $\chi = 0$ on only one of these curves, for instance the outermost (C_0 in Fig. 13). The values of χ on the remaining bounding curves are found from conditions which are a consequence of the one-valuedness of the displacement $u_z = \tau\psi(x, y)$ as a function of the co-ordinates. For, since the torsion function $\psi(x, y)$ is one-valued, the integral of its differential $d\psi$ round a closed contour must be zero. Using the relations (16.10), we therefore have

$$\begin{aligned}\oint d\psi &= \oint \left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right)\\ &= -2\oint\left(\frac{\partial\chi}{\partial x}dy - \frac{\partial\chi}{\partial y}dx\right) - 2\oint(x\,dy - y\,dx)\\ &= 0,\end{aligned}$$

or

$$\oint \frac{\partial\chi}{\partial n}dl = -S, \tag{16.13}$$

where $\partial\chi/\partial n$ is the derivative of the function χ along the outward normal to the curve, and S the area enclosed by the curve. Applying (16.13) to each of the closed curves C_1, C_2, ... , we obtain the required conditions.

Let us determine the free energy of a rod under torsion. The energy per unit volume is

$$F = \tfrac{1}{2}\sigma_{ik}u_{ik} = \sigma_{xz}u_{xz} + \sigma_{yz}u_{yz} = (\sigma_{xz}^2 + \sigma_{yz}^2)/2\mu$$

or, substituting (16.9),

$$F = 2\mu\tau^2\left[\left(\frac{\partial\chi}{\partial x}\right)^2 + \left(\frac{\partial\chi}{\partial y}\right)^2\right] \equiv 2\mu\tau^2(\mathbf{grad}\,\chi)^2,$$

where **grad** denotes the two-dimensional gradient. The torsional energy per unit length of the rod is obtained by integrating over the cross-section of the rod, i.e. it is $\frac{1}{2}C\tau^2$, where the constant $C = 4\mu\int(\mathbf{grad}\,\chi)^2\,df$, and is called the *torsional rigidity* of the rod. The total elastic energy of the rod is equal to the integral

$$F_{\text{rod}} = \tfrac{1}{2}\int C\tau^2\,dz, \tag{16.14}$$

taken along its length.

Putting

$$(\mathbf{grad}\,\chi)^2 = \text{div}(\chi\,\mathbf{grad}\,\chi) - \chi\triangle\chi = \text{div}(\chi\,\mathbf{grad}\,\chi) + \chi$$

and transforming the integral of the first term into one along the circumference of the rod, we obtain

$$C = 4\mu\oint \chi\frac{\partial\chi}{\partial n}dl + 4\mu\int\chi\,df. \tag{16.15}$$

If the cross-section is singly connected, the first term vanishes by the boundary condition $\chi = 0$, leaving

$$C = 4\mu \int \chi \, dx \, dy. \tag{16.16}$$

For a multiply connected cross-section (Fig. 13), we put $\chi = 0$ on the outer boundary C_0 and denote by χ_k the constant values of χ on the inner boundaries C_k, obtaining by (16.13)

$$C = 4\mu \sum_k \chi_k S_k + 4\mu \int \chi \, dx \, dy; \tag{16.17}$$

it should be remembered that, in integrating in the first term in (16.15), we go anti-clockwise round the contour C_0 and clockwise round all the others.

Let us consider now a more usual case of torsion, where one of the ends of the rod is held fixed and the external forces are applied only to the other end. These forces are such that they cause only a twisting of the rod, and no other deformation such as bending. In other words, they form a couple which twists the rod about its axis. The moment of this couple will be denoted by M.

We should expect that, in such a case, the torsion angle τ is constant along the rod. This can be seen, for example, from the condition that the free energy of the rod is a minimum in equilibrium. The total energy of a deformed rod is equal to the sum $F_{\text{rod}} + U$, where U is the potential energy due to the action of the external forces. Substituting in (16.14) $\tau = d\phi/dz$ and varying with respect to the angle ϕ, we find

$$\delta \tfrac{1}{2} \int C \left(\frac{d\phi}{dz}\right)^2 dz + \delta U = \int C \frac{d\phi}{dz} \frac{d\delta\phi}{dz} dz + \delta U = 0,$$

or, integrating by parts,

$$-\int C \frac{d\tau}{dz} \delta\phi \, dz + \delta U + [C\tau\delta\phi] = 0.$$

The last term on the left is the difference of the values at the limits of integration, i.e. at the ends of the rod. One of these ends, say the lower one, is fixed, so that $\delta\phi = 0$ there. The variation δU of the potential energy is minus the work done by the external forces in rotation through an angle $\delta\phi$. As we know from mechanics, the work done by a couple in such a rotation is equal to the product $M\delta\phi$ of the angle of rotation and the moment of the couple. Since there are no other external forces, $\delta U = -M\delta\phi$, and we have

$$\int C \frac{d\tau}{dz} \delta\phi \, dz + [\delta\phi(-M + C\tau)] = 0. \tag{16.18}$$

The second term on the left has its value at the upper end of the rod. In the

integral over z, the variation $\delta\phi$ is arbitrary, and so we must have

$$C\,\mathrm{d}\tau/\mathrm{d}z = 0,$$

i.e.

$$\tau = \text{constant}. \tag{16.19}$$

Thus the torsion angle is constant along the rod. The total angle of rotation of the upper end of the rod relative to the lower end is τl, where l is the length of the rod.

In equation (16.18), the second term also must be zero, and we obtain the following expression for the constant torsion angle:

$$\tau = M/C. \tag{16.20}$$

PROBLEMS

PROBLEM 1. Determine the torsional rigidity of a rod whose cross-section is a circle of radius R.

SOLUTION. The solutions of Problems 1–4 are formally identical with those of problems of the motion of a viscous fluid in a pipe of corresponding cross-section (see the last footnote to this section). The discharge Q is here represented by C.

For a rod of circular cross-section we have, taking the origin at the centre of the circle, $\chi = \frac{1}{4}(R^2-x^2-y^2)$, and the torsional rigidity is $C = \frac{1}{2}\mu\pi R^4$. For the function ψ we have, from (16.10), ψ = constant. A constant ψ, however, corresponds by (16.4) to a simple displacement of the whole rod along the z-axis, and so we can suppose that $\psi = 0$. Thus the transverse sections of a circular rod undergoing torsion remain plane.

PROBLEM 2. The same as Problem 1, but for an elliptical cross-section of semi-axes a and b.

SOLUTION. The torsional rigidity is $C = \pi\mu a^3b^3/(a^2+b^2)$. The distribution of longitudinal displacements is given by the torsion function $\psi = (b^2-a^2)xy/(b^2+a^2)$, where the co-ordinate axes coincide with those of the ellipse.

PROBLEM 3. The same as Problem 1, but for an equilateral triangular cross-section of side a.

SOLUTION. The torsional rigidity is $C = \sqrt{3}\mu a^4/80$. The torsion function is

$$\psi = y(x\sqrt{3}+y)(x\sqrt{3}-y)/6a$$

the origin being at the centre of the triangle and the x-axis along an altitude.

PROBLEM 4. The same as Problem 1, but for a rod in the form of a long thin plate (of width d and thickness $h \ll d$).

SOLUTION. The problem is equivalent to that of viscous fluid flow between plane parallel walls. The result is that $C = \frac{1}{3}\mu dh^3$.

PROBLEM 5. The same as Problem 1, but for a cylindrical pipe of internal and external radii R_1 and R_2 respectively.

SOLUTION. The function $\chi = \frac{1}{4}(R_2{}^2-r^2)$ (in polar co-ordinates) satisfies the condition (16.13) at both boundaries of the annular cross-section of the pipe. From formula (16.17) we then find $C = \frac{1}{2}\mu\pi(R_2{}^4-R_1{}^4)$.

PROBLEM 6. The same as Problem 1, but for a thin-walled pipe of arbitrary cross-section.

SOLUTION. Since the walls are thin, we can assume that χ varies through the wall thickness h, from zero on one side to χ_1 on the other, according to the linear law $\chi = \chi_1 y/h$ (y being a

co-ordinate measured through the wall). Then the condition (16.13) gives $\chi_1 L/h = S$, where L is the perimeter of the pipe cross-section and S the area which it encloses. The second term in the expression (16.17) is small compared with the first, and we obtain $C = 4hS^2\mu/L$. If the pipe is cut longitudinally along a generator, the torsional rigidity falls sharply, becoming (by the result of Problem 4) $C = \frac{1}{3}\mu Lh^3$.

§17. Bending of rods

A bent rod is stretched at some points and compressed at others. Lines on the convex side of the bent rod are extended, and those on the concave side are compressed. As with plates, there is a neutral surface in the rod, which undergoes neither extension nor compression. It separates the region of compression from the region of extension.

Let us begin by investigating a bending deformation in a small portion of the length of the rod, where the bending may be supposed slight; by this we here mean that not only the strain tensor but also the magnitudes of the displacements of points in the rod are small. We take a co-ordinate system with the origin on the neutral surface in the portion considered, and the z-axis parallel to the axis of the undeformed rod. Let the bending occur in the zx-plane.†

As in the bending of plates and the twisting of rods, the external forces on the sides of a thin bent rod are small compared with the internal stresses, and can be taken as zero in determining the boundary conditions at the sides of the rod. Thus we have everywhere on the sides of the rod $\sigma_{ik}n_k = 0$, or, since $n_z = 0$, $\sigma_{xx}n_x + \sigma_{xy}n_y = 0$, and similarly for $i = y, z$. We take a point on the circumference of a cross-section for which the normal $\mathbf{n}$ is parallel to the x-axis. There will be another such point somewhere on the opposite side of the rod. At both these points $n_y = 0$, and the above equation gives $\sigma_{xx} = 0$. Since the rod is thin, however, σ_{xx} must be small everywhere in the cross-section if it vanishes on either side. We can therefore put $\sigma_{xx} = 0$ everywhere in the rod. In a similar manner, it can be seen that all the components of the stress tensor except σ_{zz} must be zero. That is, in the bending of a thin rod only the extension (or compression) component of the internal stress tensor is large. A deformation in which only the component σ_{zz} of the stress tensor is non-zero is just a simple extension or compression (§5). Thus there is a simple extension or compression in every volume element of a bent rod. The amount of this varies, of course, from point to point in every cross-section, and so the whole rod is bent.

It is easy to determine the relative extension at any point in the rod. Let us consider an element of length dz parallel to the axis of the rod and near the origin. On bending, the length of this element becomes dz'. The only elements which remain unchanged are those which lie in the neutral surface. Let R be the radius of curvature of the neutral surface near the origin. The

---

† In a rod undergoing only small deflections we can suppose that the bending occurs in a single plane. This follows from the result of differential geometry that the deviation of a slightly bent curve from a plane (its torsion) is of a higher order of smallness than its curvature.

lengths dz and dz' can be regarded as elements of arcs of circles whose radii are respectively R and $R+x$, x being the co-ordinate of the point where dz' lies. Hence

$$dz' = \frac{R+x}{R} dz = \left(1+\frac{x}{R}\right) dz.$$

The relative extension is therefore $(dz'-dz)/dz = x/R$.

The relative extension of the element dz, however, is equal to the component u_{zz} of the strain tensor. Thus

$$u_{zz} = x/R. \tag{17.1}$$

We can now find σ_{zz} by using the relation $\sigma_{zz} = Eu_{zz}$ which holds for a simple extension. This gives

$$\sigma_{zz} = Ex/R. \tag{17.2}$$

The position of the neutral surface in a bent rod has now to be determined. This can be done from the condition that the deformation considered must be pure bending, with no general extension or compression of the rod. The total internal stress force on a cross-section of the rod must therefore be zero, i.e. the integral $\int \sigma_{zz}\, df$, taken over a cross-section, must vanish. Using the expression (17.2) for σ_{zz}, we obtain the condition

$$\int x\, df = 0. \tag{17.3}$$

We can now bring in the centre of mass of the cross-section, which is that of a uniform flat disc of the same shape. The co-ordinates of the centre of mass are, as we know, given by the integrals $\int x\, df/\int df$, $\int y\, df/\int df$. Thus the condition (17.3) signifies that, in a co-ordinate system with the origin in the neutral surface, the x co-ordinate of the centre of mass of any cross-section is zero. The neutral surface therefore passes through the centres of mass of the cross-sections of the rod.

Two components of the strain tensor besides u_{zz} are non-zero, since for a simple extension we have $u_{xx} = u_{yy} = -\sigma u_{zz}$. Knowing the strain tensor, we can easily find the displacement also:

$$u_{zz} = \partial u_z/\partial z = x/R, \qquad \partial u_x/\partial x = \partial u_y/\partial y = -\sigma x/R,$$

$$\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z} = 0, \qquad \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} = 0, \qquad \frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y} = 0.$$

Integration of these equations gives the following expressions for the components of the displacement:

$$u_x = -\frac{1}{2R}\{z^2+\sigma(x^2-y^2)\},$$
$$u_y = -\sigma xy/R, \qquad u_z = xz/R. \tag{17.4}$$

The constants of integration have been put equal to zero; this means that we "fix" the origin.

It is seen from formulae (17.4) that the points initially on a cross-section z = constant $\equiv z_0$ will be found, after the deformation, on the surface $z = z_0 + u_z = z_0(1+x/R)$. We see that, in the approximation used, the cross-sections remain plane but are turned through an angle relative to their initial positions. The shape of the cross-section changes, however; for example, when a rod of rectangular cross-section (sides a, b) is bent, the sides $y = \pm\frac{1}{2}b$ of the cross-section become $y = \pm\frac{1}{2}b + u_y = \pm\frac{1}{2}b(1-\sigma x/R)$, i.e. no longer parallel but still straight. The sides $x = \pm\frac{1}{2}a$, however, are bent into the parabolic curves

$$x = \pm\tfrac{1}{2}a + u_x = \pm\tfrac{1}{2}a - \frac{1}{2R}[z_0{}^2 + \sigma(\tfrac{1}{4}a^2 - y^2)]$$

(Fig. 14).

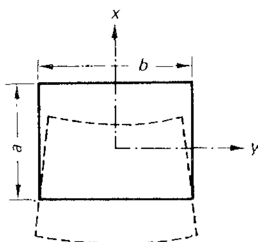


FIG. 14

The free energy per unit volume of the rod is

$$\tfrac{1}{2}\sigma_{ik}u_{ik} = \tfrac{1}{2}\sigma_{zz}u_{zz} = \tfrac{1}{2}Ex^2/R^2.$$

Integrating over the cross-section of the rod, we have

$$\tfrac{1}{2}(E/R^2)\int x^2\, \mathrm{d}f. \tag{17.5}$$

This is the free energy per unit length of a bent rod. The radius of curvature R is that of the neutral surface. However, since the rod is thin, R can here be regarded, to the same approximation, as the radius of curvature of the bent rod itself, regarded as a line (often called an "elastic line").

In the expression (17.5) it is convenient to introduce the moment of inertia of the cross-section. The moment of inertia about the y-axis in its plane is defined as

$$I_y = \int x^2\, \mathrm{d}f, \tag{17.6}$$

analogously to the ordinary moment of inertia, but with the surface element df instead of the mass element. Then the free energy per unit length of the rod can be written

$$\tfrac{1}{2}EI_y/R^2. \tag{17.7}$$

We can also determine the moment of the internal stress forces on a given cross-section of the rod (the *bending moment*). A force $\sigma_{zz}\, \mathrm{d}f = (xE/R)\, \mathrm{d}f$ acts in the z-direction on the surface element $\mathrm{d}f$ of the cross-section. Its moment about the y-axis is $x\sigma_{zz}\, \mathrm{d}f$. Hence the total moment of the forces about this axis is

$$M_y = (E/R) \int x^2\, \mathrm{d}f = EI_y/R. \tag{17.8}$$

Thus the curvature $1/R$ of the elastic line is proportional to the bending moment on the cross-section concerned.

The magnitude of I_y depends on the direction of the y-axis in the cross-sectional plane. It is convenient to express I_y in terms of the principal moments of inertia. If θ is the angle between the y-axis and one of the principal axes of inertia in the cross-section, we know from mechanics that

$$I_y = I_1 \cos^2\theta + I_2 \sin^2\theta, \tag{17.9}$$

where I_1 and I_2 are the principal moments of inertia. The planes through the z-axis and the principal axes of inertia are called the *principal planes of bending*.

If, for example, the cross-section is rectangular (with sides a, b), its centre of mass is at the centre of the rectangle, and the principal axes of inertia are parallel to the sides. The principal moments of inertia are

$$I_1 = a^3b/12, \qquad I_2 = ab^3/12. \tag{17.10}$$

For a circular cross-section of radius R, the centre of mass is at the centre of the circle, and the principal axes are arbitrary. The moment of inertia about any axis lying in the cross-section and passing through the centre is

$$I = \tfrac{1}{4}\pi R^4. \tag{17.11}$$

§18. The energy of a deformed rod

In §17 we have discussed only a small portion of the length of a bent rod. In going on to investigate the deformation throughout the rod, we must begin by finding a suitable method of describing this deformation. It is important to note that, when a rod undergoes large bending deflections,† there is in general a twisting of it as well, so that the resulting deformation is a combination of pure bending and torsion.

To describe the deformation, it is convenient to proceed as follows. We divide the rod into infinitesimal elements, each of which is bounded by two adjacent cross-sections. For each such element we use a co-ordinate system ξ, η, ζ, so chosen that all the systems are parallel in the undeformed state, and their ζ-axes are parallel to the axis of the rod. When the rod is bent, the

† By this, it should be remembered, we mean that the vector **u** is not small, but the strain tensor is still small.

co-ordinate system in each element is rotated, and in general differently in different elements. Any two adjacent systems are rotated through an infinitesimal relative angle.

Let $\mathrm{d}\boldsymbol{\phi}$ be the vector of the angle of relative rotation of two systems at a distance $\mathrm{d}l$ apart along the rod (we know that an infinitesimal angle of rotation can be regarded as a vector parallel to the axis of rotation; its components are the angles of rotation about each of the three axes of co-ordinates).

To describe the deformation, we use the vector

$$\boldsymbol{\Omega} = \mathrm{d}\boldsymbol{\phi}/\mathrm{d}l, \tag{18.1}$$

which gives the "rate" of rotation of the co-ordinate axes along the rod. If the deformation is a pure torsion, the co-ordinate system rotates only about the axis of the rod, i.e. about the ζ-axis. In this case, therefore, the vector $\boldsymbol{\Omega}$ is parallel to the axis of the rod, and is just the torsion angle τ used in §16. Correspondingly, in the general case of an arbitrary deformation we can call the component Ω_ζ of the vector $\boldsymbol{\Omega}$ the *torsion angle*. For a pure bending of the rod in a single plane, on the other hand, the vector $\boldsymbol{\Omega}$ has no component Ω_ζ, i.e. it lies in the $\xi\eta$-plane at each point. If we take the plane of bending as the $\xi\zeta$-plane, then the rotation is about the η-axis at every point, i.e. $\boldsymbol{\Omega}$ is parallel to the η-axis.

We take a unit vector $\mathbf{t}$ tangential to the rod (regarded as an elastic line). The derivative $\mathrm{d}\mathbf{t}/\mathrm{d}l$ is the curvature vector of the line; its magnitude is $1/R$, where R is the radius of curvature,† and its direction is that of the principal normal to the curve. The change in a vector due to an infinitesimal rotation is equal to the vector product of the rotation vector and the vector itself. Hence the change in the vector $\mathbf{t}$ between two neighbouring points of the elastic line is given by $\mathrm{d}\mathbf{t} = \mathrm{d}\boldsymbol{\phi} \times \mathbf{t}$, or, dividing by $\mathrm{d}l$,

$$\mathrm{d}\mathbf{t}/\mathrm{d}l = \boldsymbol{\Omega} \times \mathbf{t}. \tag{18.2}$$

Multiplying this equation vectorially by $\mathbf{t}$, we have

$$\boldsymbol{\Omega} = \mathbf{t} \times \mathrm{d}\mathbf{t}/\mathrm{d}l + \mathbf{t}(\mathbf{t} \cdot \boldsymbol{\Omega}). \tag{18.3}$$

The direction of the tangent vector at any point is the same as that of the ζ-axis at that point. Hence $\mathbf{t} \cdot \boldsymbol{\Omega} = \Omega_\zeta$. Using the unit vector $\mathbf{n}$ along the principal normal ($\mathbf{n} = R\,\mathrm{d}\mathbf{t}/\mathrm{d}l$), we can therefore put

$$\boldsymbol{\Omega} = \mathbf{t} \times \mathbf{n}/R + \mathbf{t}\Omega_\zeta. \tag{18.4}$$

The first term on the right is a vector with two components Ω_ξ, Ω_η. The unit vector $\mathbf{t} \times \mathbf{n}$ is the binormal unit vector. Thus the components Ω_ξ, Ω_η form a vector along the binormal to the rod, whose magnitude equals the curvature $1/R$.

† It may be recalled that any curve in space is characterised at each point by a curvature and a torsion. This torsion (which we shall not use) should not be confused with the torsional deformation, which is a twisting of a rod about its axis.

By using the vector $\mathbf{\Omega}$ to characterise the deformation and ascertaining its properties, we can derive an expression for the elastic free energy of a bent rod. The elastic energy per unit length of the rod is a quadratic function of the deformation, i.e., in this case, a quadratic function of the components of the vector $\mathbf{\Omega}$. It is easy to see that there can be no terms in this quadratic form proportional to $\Omega_\xi\Omega_\zeta$ and $\Omega_\eta\Omega_\zeta$. For, since the rod is uniform along its length, all quantities, and in particular the energy, must remain constant when the direction of the positive ζ-axis is reversed, i.e. when ζ is replaced by $-\zeta$, whereas the products mentioned change sign.

For $\Omega_\xi = \Omega_\eta = 0$ we have a pure torsion, and the expression for the energy must be that obtained in §16. Thus the term in Ω_ζ^2 in the free energy is $\frac{1}{2}C\Omega_\zeta^2$.

Finally, the terms quadratic in Ω_ξ and Ω_η can be obtained by starting from the expression (17.7) for the energy of a slightly bent short section of the rod. Let us suppose that the rod is only slightly bent. We take the $\xi\zeta$-plane as the plane of bending, so that the component Ω_ξ is zero; there is also no torsion in a slight bending. The expression for the energy must then be that given by (17.7), i.e. $\frac{1}{2}EI_\eta/R^2$. We have seen, however, that $1/R^2$ is the square of the two-dimensional vector $(\Omega_\xi, \Omega_\eta)$. Hence the energy must be of the form $\frac{1}{2}EI_\eta\Omega_\eta^2$. For an arbitrary choice of the ξ and η axes this expression becomes, as we know from mechanics,

$$\tfrac{1}{2}E(I_{\eta\eta}\Omega_\eta^2+2I_{\eta\xi}\Omega_\eta\Omega_\xi+I_{\xi\xi}\Omega_\xi^2),$$

where $I_{\eta\eta}$, $I_{\eta\xi}$, $I_{\xi\xi}$ are the components of the inertia tensor for the cross-section of the rod. It is convenient to take the ξ and η axes to coincide with the principal axes of inertia. We then have simply $\frac{1}{2}E(I_1\Omega_\xi^2+I_2\Omega_\eta^2)$, where I_1, I_2 are the principal moments of inertia. Since the coefficients of Ω_ξ^2 and Ω_η^2 are constants, the resulting expression must be valid for large deflections also.

Finally, integrating over the length of the rod, we obtain the following expression for the elastic free energy of a bent rod:

$$F_{\text{rod}} = \int \{\tfrac{1}{2}I_1E\Omega_\xi^2+\tfrac{1}{2}I_2E\Omega_\eta^2+\tfrac{1}{2}C\Omega_\zeta^2\}\mathrm{d}l. \tag{18.5}$$

Next, we can express in terms of $\mathbf{\Omega}$ the moment of the forces acting on a cross-section of the rod. This is easily done by again using the results previously obtained for pure torsion and pure bending. In pure torsion, the moment of the forces about the axis of the rod is $C\tau$. Hence we conclude that, in the general case, the moment M_ζ about the ζ-axis must be $C\Omega_\zeta$. Next, in a slight deflection in the $\xi\zeta$-plane, the moment about the η-axis is EI_2/R. In such a bending, however, the vector $\mathbf{\Omega}$ is along the η-axis, so that $1/R$ is just the magnitude of $\mathbf{\Omega}$, and $EI_2/R = EI_2\Omega$. Hence we conclude that, in the general case, we must have $M_\xi = EI_1\Omega_\xi$, $M_\eta = EI_2\Omega_\eta$ (the ξ and η axes being along the principal axes of inertia in the cross-section). Thus

the components of the moment vector **M** are

$$M_\xi = EI_1\Omega_\xi, \qquad M_\eta = EI_2\Omega_\eta, \qquad M_\zeta = C\Omega_\zeta. \tag{18.6}$$

The elastic energy (18.5), expressed in terms of the moment of the forces, is

$$F_{\text{rod}} = \int\left\{\frac{M_\xi^2}{2I_1E}+\frac{M_\eta^2}{2I_2E}+\frac{M_\zeta^2}{2C}\right\}\mathrm{d}l. \tag{18.7}$$

An important case of the bending of rods is that of a slight bending, in which the deviation from the initial position is everywhere small compared with the length of the rod. In this case torsion can be supposed absent, and we can put $\Omega_\zeta = 0$, so that (18.4) gives simply

$$\mathbf{\Omega} = \mathbf{t}\times\mathbf{n}/R = \mathbf{t}\times\mathrm{d}\mathbf{t}/\mathrm{d}l. \tag{18.8}$$

We take a co-ordinate system x, y, z fixed in space, with the z axis along the axis of the undeformed rod (instead of the system ξ, η, ζ for each point in the rod), and denote by X, Y the co-ordinates x, y for points on the elastic line; X and Y give the displacement of points on the line from their positions before the deformation.

Since the bending is only slight, the tangent vector $\mathbf{t}$ is almost parallel to the z-axis, and the difference in direction can be approximately neglected. The unit tangent vector is the derivative $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}l$ of the radius vector $\mathbf{r}$ of a point on the curve with respect to its length. Hence

$$\mathrm{d}\mathbf{t}/\mathrm{d}l = \mathrm{d}^2\mathbf{r}/\mathrm{d}l^2 \approx \mathrm{d}^2\mathbf{r}/\mathrm{d}z^2;$$

the derivative with respect to the length can be approximately replaced by the derivative with respect to z. In particular, the x and y components of this vector are respectively $\mathrm{d}^2X/\mathrm{d}z^2$ and $\mathrm{d}^2Y/\mathrm{d}z^2$. The components Ω_ξ, Ω_η are, to the same accuracy, equal to Ω_x, Ω_y, and we have from (18.8)

$$\Omega_\xi = -\mathrm{d}^2Y/\mathrm{d}z^2, \qquad \Omega_\eta = \mathrm{d}^2X/\mathrm{d}z^2. \tag{18.9}$$

Substituting these expressions in (18.5), we obtain the elastic energy of a slightly bent rod in the form

$$F_{\text{rod}} = \tfrac{1}{2}E\int\left\{I_1\left(\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right)^2+I_2\left(\frac{\mathrm{d}^2X}{\mathrm{d}z^2}\right)^2\right\}\mathrm{d}z. \tag{18.10}$$

Here I_1 and I_2 are the moments of inertia about the axes of x and y respectively, which are the principal axes of inertia.

In particular, for a rod of circular cross-section, $I_1 = I_2 = I$, and the integrand is just the sum of the squared second derivatives, which in the approximation considered is the square of the curvature:

$$\left(\frac{\mathrm{d}^2X}{\mathrm{d}z^2}\right)^2+\left(\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right)^2 = \frac{1}{R^2}.$$

Hence formula (18.10) can be plausibly generalised to the case of slight bending of a circular rod having any shape (not necessarily straight) in its undeformed state. To do so, we must write the bending energy as

$$F_{\mathrm{rod}} = \tfrac{1}{2}EI \int \left(\frac{1}{R} - \frac{1}{R_0}\right)^2 \mathrm{d}z, \tag{18.11}$$

where R_0 is the radius of curvature at any point of the undeformed rod. This expression has a minimum, as it should, in the undeformed state ($R = R_0$), and for $R_0 \to \infty$ it becomes formula (18.10).

§19. The equations of equilibrium of rods

We can now derive the equations of equilibrium for a bent rod. We again consider an infinitesimal element bounded by two adjoining cross-sections of the rod, and calculate the total force acting on it. We denote by $\mathbf{F}$ the resultant internal stress on a cross-section.† The components of this vector are the integrals of $\sigma_{i\zeta}$ over the cross-section:

$$F_i = \int \sigma_{i\zeta}\, \mathrm{d}f. \tag{19.1}$$

If we regard the two adjoining cross-sections as the ends of the element, a force $\mathbf{F}+\mathrm{d}\mathbf{F}$ acts on the upper end, and $-\mathbf{F}$ on the lower end; the sum of these is the differential $\mathrm{d}\mathbf{F}$. Next, let $\mathbf{K}$ be the external force on the rod per unit length. Then an external force $\mathbf{K}\,\mathrm{d}l$ acts on the element of length $\mathrm{d}l$. The resultant of the forces on the element is therefore $\mathrm{d}\mathbf{F}+\mathbf{K}\,\mathrm{d}l$. This must be zero in equilibrium. Thus we have

$$\mathrm{d}\mathbf{F}/\mathrm{d}l = -\mathbf{K}. \tag{19.2}$$

A second equation is obtained from the condition that the total moment of the forces on the element is zero. Let $\mathbf{M}$ be the moment of the internal stresses on the cross-section. This is the moment about a point (the origin) which lies in the plane of the cross-section; its components are given by formulae (18.6). We shall calculate the total moment, on the element considered, about a point O lying in the plane of its upper end. Then the internal stresses on this end give a moment $\mathbf{M}+\mathrm{d}\mathbf{M}$. The moment about O of the internal stresses on the lower end of the element is composed of the moment $-\mathbf{M}$ of those forces about the origin O' in the plane of the lower end and the moment about O of the total force $-\mathbf{F}$ on that end. This latter moment is $-\mathrm{d}\mathbf{l}\times-\mathbf{F}$, where $\mathrm{d}\mathbf{l}$ is the vector of the element of length of the rod between O' and O. The moment due to the external forces $\mathbf{K}$ is of a higher order of smallness. Thus the total moment acting on the element considered is $\mathrm{d}\mathbf{M}+\mathrm{d}\mathbf{l}\times\mathbf{F}$. In equilibrium, this must be zero:

$$\mathrm{d}\mathbf{M}+\mathrm{d}\mathbf{l}\times\mathbf{F} = 0.$$

† This notation will not lead to any confusion with the free energy, which does not appear in §§19–21.

Dividing this equation by dl and using the fact that $d\mathbf{l}/dl = \mathbf{t}$ is the unit vector tangential to the rod (regarded as a line), we have

$$d\mathbf{M}/dl = \mathbf{F}\times\mathbf{t}. \tag{19.3}$$

Equations (19.2) and (19.3) form a complete set of equilibrium equations for a rod bent in any manner.

If the external forces on the rod are concentrated, i.e. applied only at isolated points of the rod, the equilibrium equations at all other points are much simplified. For $\mathbf{K} = 0$ we have from (19.2)

$$\mathbf{F} = \text{constant}, \tag{19.4}$$

i.e. the stress resultant is constant along any portion of the rod between points where forces are applied. The values of the constant are found from the fact that the difference $\mathbf{F}_2 - \mathbf{F}_1$ of the forces at two points 1 and 2 is

$$\mathbf{F}_2 - \mathbf{F}_1 = -\Sigma\mathbf{K}, \tag{19.5}$$

where the sum is over all forces applied to the segment of the rod between the two points. It should be noticed that, in the difference $\mathbf{F}_2 - \mathbf{F}_1$, the point 2 is further from the point from which l is measured than is the point 1; this is important in determining the signs in equation (19.5). In particular, if only one concentrated force $\mathbf{f}$ acts on the rod, and is applied at its free end, then $\mathbf{F}$ = constant = $\mathbf{f}$ at all points of the rod.

The second equilibrium equation (19.3) is also simplified. Putting $\mathbf{t} = d\mathbf{l}/dl = d\mathbf{r}/dl$ (where $\mathbf{r}$ is the radius vector from any fixed point to the point considered) and integrating, we obtain

$$\mathbf{M} = \mathbf{F}\times\mathbf{r} + \text{constant}, \tag{19.6}$$

since $\mathbf{F}$ is constant.

If concentrated forces also are absent, and the rod is bent by the application of concentrated moments, i.e. of concentrated couples, then $\mathbf{F}$ = constant at all points of the rod, while $\mathbf{M}$ is discontinuous at points where couples are applied, the discontinuity being equal to the moment of the couple.

Let us consider also the boundary conditions at the ends of a bent rod. Various cases are possible.

The end of the rod is said to be *clamped* (Fig. 4a, §12) if it cannot move either longitudinally or transversely, and moreover its direction (i.e. the direction of the tangent to the rod) cannot change. In this case the boundary conditions are that the co-ordinates of the end of the rod and the unit tangential vector $\mathbf{t}$ there are given. The reaction force and moment exerted on the rod by the clamp are determined by solving the equations.

The opposite case is that of a free end, whose position and direction are arbitrary. In this case the boundary conditions are that the force $\mathbf{F}$ and moment $\mathbf{M}$ must be zero at the end of the rod.†

† If a concentrated force $\mathbf{f}$ is applied to the free end of the rod, the boundary condition is $\mathbf{F} = \mathbf{f}$, not $\mathbf{F} = 0$.

If the end of the rod is fixed to a hinge, it cannot be displaced, but its direction can vary. In this case the moment of the forces on the freely turning end must be zero.

Finally, if the rod is supported (Fig. 4b), it can slide at the point of support but cannot undergo transverse displacements. In this case the direction $\mathbf{t}$ of the rod at the support and the point on the rod at which it is supported are unknown. The moment of the forces at the point of support must be zero, since the rod can turn freely, and the force $\mathbf{F}$ at that point must be perpendicular to the rod; a longitudinal force would cause a further sliding of the rod at this point.

The boundary conditions for other modes of fixing the rod can easily be established in a similar manner. We shall not pause to add to the typical examples already given.

It was mentioned at the beginning of §18 that a rod of arbitrary cross-section undergoing large deflections is in general twisted also, even if no external twisting moment is applied to the rod. An exception occurs when a rod is bent in one of its principal planes, in which case there is no torsion. For a rod of circular cross-section no torsion results for any bending (if there is no external twisting moment, of course). This can be seen as follows. The twisting is given by the component $\Omega_\zeta = \mathbf{\Omega}\cdot\mathbf{t}$ of the vector $\mathbf{\Omega}$. Let us calculate the derivative of this along the rod. To do so, we use the fact that $\Omega_\zeta = M_\zeta/C$:

$$\frac{\mathrm{d}}{\mathrm{d}l}(\mathbf{M}\cdot\mathbf{t}) = C\frac{\mathrm{d}\Omega_\zeta}{\mathrm{d}l} = \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}l}\cdot\mathbf{t}+\mathbf{M}\cdot\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}l}.$$

Substituting (19.3), we see that the first term is zero, so that

$$C\mathrm{d}\Omega_\zeta/\mathrm{d}l = \mathbf{M}\cdot\mathrm{d}\mathbf{t}/\mathrm{d}l.$$

For a rod of circular cross-section, $I_1 = I_2 \equiv I$; by (18.3) and (18.6), we can therefore write $\mathbf{M}$ in the form

$$\mathbf{M} = EI\mathbf{t}\times\mathrm{d}\mathbf{t}/\mathrm{d}l+\mathbf{t}C\Omega_\zeta. \tag{19.7}$$

Multiplying by $\mathrm{d}\mathbf{t}/\mathrm{d}l$, we have zero on the right-hand side, so that

$$\mathrm{d}\Omega_\zeta/\mathrm{d}l = 0,$$

whence

$$\Omega_\zeta = \text{constant}, \tag{19.8}$$

i.e. the torsion angle is constant along the rod. If no twisting moments are applied to the ends of the rod, then Ω_ζ is zero at the ends, and there is no torsion anywhere in the rod.

For a rod of circular cross-section, we can therefore put for pure bending

$$M = EI\mathbf{t}\times\mathrm{d}\mathbf{t}/\mathrm{d}l = EI\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}l^2}. \tag{19.9}$$

Substituting this in (19.3), we obtain the equation for pure bending of a circular rod:

$$EI\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}\times\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}l^3} = \mathbf{F}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}. \tag{19.10}$$

PROBLEMS

PROBLEM 1. Reduce to quadratures the problem of determining the shape of a rod of circular cross-section bent in one plane by concentrated forces.

SOLUTION. Let us consider a portion of the rod lying between points where the forces are applied; on such a portion $\mathbf{F}$ is constant. We take the plane of the bent rod as the xy plane, with the y-axis parallel to the force $\mathbf{F}$, and introduce the angle θ between the tangent to the rod and the y-axis. Then $\mathrm{d}x/\mathrm{d}l = \sin\theta$, $\mathrm{d}y/\mathrm{d}l = \cos\theta$, where x, y are the co-ordinates of a point on the rod. Expanding the vector products in (19.10), we obtain the following equation for θ as a function of the arc length l: $EI\mathrm{d}^2\theta/\mathrm{d}l^2 - F\sin\theta = 0$. A first integration gives $\frac{1}{2}EI(\mathrm{d}\theta/\mathrm{d}l)^2 + F\cos\theta = c_1$, and

$$l = \pm\sqrt{(\tfrac{1}{2}EI)}\int\frac{\mathrm{d}\theta}{\sqrt{(c_1 - F\cos\theta)}} + c_2. \tag{1}$$

The function $\theta(l)$ can be obtained in terms of elliptic functions. The co-ordinates

$$x = \int\sin\theta\,\mathrm{d}l, \qquad y = \int\cos\theta\,\mathrm{d}l$$

are

$$x = \pm\sqrt{[2EI(c_1 - F\cos\theta)/F^2]} + \text{constant},$$

$$y = \pm\sqrt{(\tfrac{1}{2}EI)}\int\frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{(c_1 - F\cos\theta)}} + \text{constant}. \tag{2}$$

The moment $\mathbf{M}$ (19.9) is parallel to the z-axis, and its magnitude is $M = EI\mathrm{d}\theta/\mathrm{d}l$.

PROBLEM 2. Determine the shape of a bent rod with one end clamped and the other under a force $\mathbf{f}$ perpendicular to the original direction of the rod (Fig. 15).

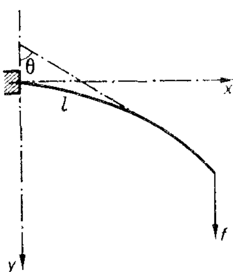


FIG. 15

SOLUTION. We have $\mathbf{F}$ = constant = $\mathbf{f}$ everywhere on the rod. At the clamped end ($l = 0$), $\theta = \frac{1}{2}\pi$, and at the free end ($l = L$, the length of the rod) $M = 0$, i.e. $\theta' = 0$. Putting $\theta(L) \equiv \theta_0$, we have in (1), Problem 1, $c_1 = f\cos\theta_0$ and

$$l = \sqrt{(\tfrac{1}{2}EI/f)}\int_{\theta}^{\frac{1}{2}\pi}\frac{\mathrm{d}\theta}{\sqrt{(\cos\theta_0 - \cos\theta)}}.$$

Hence we obtain the equation for θ_0:

$$L = \sqrt{(\tfrac{1}{2}EI/f)} \int\limits_{\theta_0}^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(\cos\theta_0 - \cos\theta)}}.$$

The shape of the rod is given by

$$x = \sqrt{(2EI/f)}[\sqrt{(\cos\theta_0)} - \sqrt{(\cos\theta_0 - \cos\theta)}],$$

$$y = \sqrt{(EI/2f)} \int\limits_{\theta}^{\frac{1}{2}\pi} \frac{\cos\theta \, d\theta}{\sqrt{(\cos\theta_0 - \cos\theta)}}.$$

PROBLEM 3. The same as Problem 2, but for a force **f** parallel to the original direction of the rod.

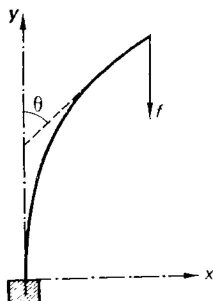


FIG. 16

SOLUTION. We have **F** = −**f**; the co-ordinate axes are taken as shown in Fig. 16. The boundary conditions are $\theta = 0$ for $l = 0$, $\theta' = 0$ for $l = L$. Then

$$l = \sqrt{(\tfrac{1}{2}EI/f)} \int\limits_{0}^{\theta} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}},$$

where

$$\theta_0 = \theta(L)$$

is given by

$$L = \sqrt{(\tfrac{1}{2}EI/f)} \int\limits_{0}^{\theta_0} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}.$$

For x and y we obtain

$$x = \sqrt{(2EI/f)}[\sqrt{(1 - \cos\theta_0)} - \sqrt{(\cos\theta - \cos\theta_0)}],$$

$$y = \sqrt{(EI/2f)} \int\limits_{0}^{\theta} \frac{\cos\theta \, d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}.$$

For a small deflection, $\theta_0 \ll 1$, and we can write

$$L \approx \sqrt{(EI/f)} \int_0^{\theta_0} \frac{d\theta}{\sqrt{(\theta_0^2 - \theta^2)}} = \tfrac{1}{2}\pi\sqrt{(EI/f)},$$

i.e. θ_0 does not appear. This shows that, in accordance with the result of §21, Problem 3, the solution in question exists only for $f \geqslant \pi^2 EI/4L^2$, i.e. when the rectilinear shape ceases to be stable.

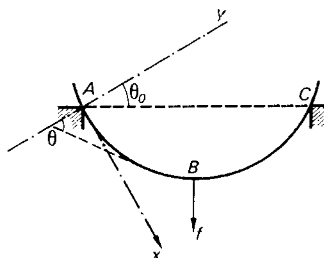


FIG. 17

PROBLEM 4. The same as Problem 2, but for the case where both ends of the rod are supported and a force **f** is applied at its centre. The distance between the supports is L_0.

SOLUTION. We take the co-ordinate axes as shown in Fig. 17. The force **F** is constant on each of the segments AB and BC, and on each is perpendicular to the direction of the rod at the point of support A or C. The difference between the values of **F** on AB and BC is **f**, and so we conclude that, on AB, $F \sin \theta_0 = -\tfrac{1}{2}f$, where θ_0 is the angle between the y-axis and the line AC. At the point A ($l = 0$) we have the conditions $\theta = \tfrac{1}{2}\pi$ and $M = 0$, i.e. $\theta' = 0$, so that on AB

$$l = \sqrt{\frac{EI \sin\theta_0}{f}} \int_\theta^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{\cos\theta}}, \qquad x = 2\sqrt{\frac{EI \sin\theta_0 \cos\theta}{f}},$$

$$y = \sqrt{\frac{EI \sin\theta_0}{f}} \int_\theta^{\frac{1}{2}\pi} \sqrt{\cos\theta}\, d\theta.$$

The angle θ_0 is determined from the condition that the projection of AB on the straight line AC must be $\tfrac{1}{2}L_0$, whence

$$\tfrac{1}{2}L_0 = \sqrt{\frac{EI \sin\theta_0}{f}} \int_{\theta_0}^{\frac{1}{2}\pi} \frac{\cos(\theta - \theta_0)}{\sqrt{\sin\theta}}\, d\theta.$$

For some value θ_0 lying between 0 and $\tfrac{1}{2}\pi$ the derivative $df/d\theta_0$ (f being regarded as a function of θ_0) passes through zero to positive values. A further decrease in θ_0, i.e. increase in the deflection, would mean a decrease in f. This means that the solution found here becomes unstable, the rod collapsing between the supports.

PROBLEM 5. Reduce to quadratures the problem of three-dimensional bending of a rod under the action of concentrated forces.

SOLUTION. Let us consider a segment of the rod between points where forces are applied, on which $\mathbf{F}$ = constant. Integrating (19.10), we obtain

$$EI\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}l^2} = \mathbf{F}\times\mathbf{r}+c\mathbf{F}; \tag{1}$$

the constant of integration has been written as a vector $c\mathbf{F}$ parallel to $\mathbf{F}$, since, by appropriately choosing the origin, i.e. by adding a constant vector to $\mathbf{r}$, we can eliminate any vector perpendicular to $\mathbf{F}$. Multiplying (1) scalarly and vectorially by $\mathbf{r}'$ (the prime denoting differentiation with respect to l), and using the fact that $\mathbf{r}'\cdot\mathbf{r}'' = 0$ (since $\mathbf{r}'^2 = 1$), we obtain $\mathbf{F}\cdot\mathbf{r}\times\mathbf{r}'+c\mathbf{F}\cdot\mathbf{r}' = 0$, $EI\mathbf{r}'' = (\mathbf{F}\times\mathbf{r})\times\mathbf{r}'+c\mathbf{F}\times\mathbf{r}'$. In components (with the z-axis parallel to $\mathbf{F}$) we obtain $(xy'-yx')+cz' = 0$, $EIz'' = -F(xx'+yy')$. Using cylindrical polar coordinates r, ϕ, z, we have

$$r^2\phi'+cz' = 0, \qquad EIz'' = -Frr'. \tag{2}$$

The second of these gives

$$z' = F(A-r^2)/2EI, \tag{3}$$

where A is a constant. Combining (2) and (3) with the identity $r'^2+r^2\phi'^2+z'^2 = 1$, we find

$$\mathrm{d}l = \frac{r\,\mathrm{d}r}{\sqrt{[r^2-(r^2+c^2)(A-r^2)^2F^2/4E^2I^2]}},$$

and then (2) and (3) give

$$z = \frac{F}{2EI}\int\frac{(A-r^2)r\,\mathrm{d}r}{\sqrt{[r^2-F^2(r^2+c^2)(A-r^2)^2/4E^2I^2]}},$$

$$\phi = -\frac{cF}{2EI}\int\frac{(A-r^2)\,\mathrm{d}r}{r\sqrt{[r^2-F^2(r^2+c^2)(A-r^2)^2/4E^2I^2]}}$$

which gives the shape of the bent rod.

PROBLEM 6. A rod of circular cross-section is subjected to torsion (with torsion angle τ) and twisted into a spiral. Determine the force and moment which must be applied to the ends of the rod to keep it in this state.

SOLUTION. Let R be the radius of the cylinder on whose surface the spiral lies (and along whose axis we take the z-direction) and α the angle between the tangent to the spiral and a plane perpendicular to the z-axis; the pitch h of the spiral is related to α and R by $h = 2\pi R\tan\alpha$. The equation of the spiral is $x = R\cos\phi$, $y = R\sin\phi$, $z = \phi R\tan\alpha$, where ϕ is the angle of rotation about the z-axis. The element of length is $\mathrm{d}l = (R/\cos\alpha)\mathrm{d}\phi$. Substituting these expressions in (19.7), we calculate the components of the vector $\mathbf{M}$, and then the force $\mathbf{F}$ from formula (19.3); $\mathbf{F}$ is constant everywhere on the rod. The result is that the force $\mathbf{F}$ is parallel to the z-axis and its magnitude is $F = F_z = (C\tau/R)\sin\alpha-(EI/R^2)\cos^2\alpha\sin\alpha$. The moment $\mathbf{M}$ has a z-component $M_z = C\tau\sin\alpha+(EI/R)\cos^3\alpha$ and a ϕ-component, along the tangent to the cross-section of the cylinder, $M_\phi = FR$.

PROBLEM 7. Determine the form of a flexible wire (whose resistance to bending can be neglected in comparison with its resistance to stretching) suspended at two points and in a gravitational field.

SOLUTION. We take the plane of the wire as the xy-plane, with the y-axis vertically downwards. In equation (19.3) we can neglect the term $\mathrm{d}\mathbf{M}/\mathrm{d}l$, since $\mathbf{M}$ is proportional to EI. Then $\mathbf{F}\times\mathbf{t} = 0$, i.e. $\mathbf{F}$ is parallel to $\mathbf{t}$ at every point, and we can put $\mathbf{F} = F\mathbf{t}$. Equation (19.2) then gives

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(F\frac{\mathrm{d}x}{\mathrm{d}l}\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}l}\left(F\frac{\mathrm{d}y}{\mathrm{d}l}\right) = q,$$

where q is the weight of the wire per unit length; hence $F\,\mathrm{d}x/\mathrm{d}l = c$, $F\,\mathrm{d}y/\mathrm{d}l = ql$, and so $F = \sqrt{(c^2+q^2l^2)}$, so that $\mathrm{d}x/\mathrm{d}l = A/\sqrt{(A^2+l^2)}$, $\mathrm{d}y/\mathrm{d}l = l/\sqrt{(A^2+l^2)}$, where $A = c/q$. Integration gives $x = A\sinh^{-1}(l/A)$, $y = \sqrt{(A^2+l^2)}$, whence $y = A\cosh(x/A)$, i.e. the wire takes the form of a catenary. The choice of origin and the constant A are determined by the fact that the curve must pass through the two given points and have a given length.

§20. Small deflections of rods

The equations of equilibrium are considerably simplified in the important case of small deflections of rods. This case holds if the direction of the vector **t** tangential to the rod varies only slowly along its length, i.e. the derivative $\mathrm{d}\mathbf{t}/\mathrm{d}l$ is small. In other words, the radius of curvature of the bent rod is everywhere large compared with the length of the rod. In practice, this condition amounts to requiring that the transverse deflection of the rod is small compared with its length. It should be emphasised that the deflection need not be small compared with the thickness of the rod, as it had to be in the approximate theory of small deflections of plates given in §§11–12.†

Differentiating (19.3) with respect to the length, we have

$$\frac{\mathrm{d}^2\mathbf{M}}{\mathrm{d}l^2} = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}l}\times\mathbf{t}+\mathbf{F}\times\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}l}. \tag{20.1}$$

The second term contains the small quantity $\mathrm{d}\mathbf{t}/\mathrm{d}l$, and so can usually be neglected (some exceptional cases are discussed below). Substituting in the first term $\mathrm{d}\mathbf{F}/\mathrm{d}l = -\mathbf{K}$, we obtain the equation of equilibrium in the form

$$\mathrm{d}^2\mathbf{M}/\mathrm{d}l^2 = \mathbf{t}\times\mathbf{K}. \tag{20.2}$$

We write this equation in components, substituting in it from (18.6) and (18.9)

$$M_x = -EI_1Y'', \qquad M_y = EI_2X'', \qquad M_z = 0, \tag{20.3}$$

where the prime denotes differentiation with respect to z. The unit vector **t** may be supposed to be parallel to the z-axis. Then (20.2) gives

$$EI_2X^{(\mathrm{iv})}-K_x = 0, \qquad EI_1Y^{(\mathrm{iv})}-K_y = 0. \tag{20.4}$$

These equations give the deflections X and Y as functions of z, i.e. the shape of a slightly bent rod.

The stress resultant **F** on a cross-section of the rod can also be expressed in terms of the derivatives of X and Y. Substituting (20.3) in (19.3), we obtain

$$F_x = -EI_2X''', \qquad F_y = -EI_1Y'''. \tag{20.5}$$

We see that the second derivatives give the moment of the internal stresses, while the third derivatives give the stress resultant. The force (20.5) is called the *shearing force*. If the bending is due to concentrated forces, the shearing force is constant along each segment of the rod between points

† We shall not give the complex theory of the bending of rods which are not straight when undeformed, but only consider one simple example (see Problems 8 and 9).

where forces are applied, and has a discontinuity at each of these points equal to the force applied there.

The quantities EI_2 and EI_1 are called the *flexural rigidities* of the rod in the xz and yz planes respectively.†

If the external forces applied to the rod act in one plane, the bending takes place in one plane, though not in general the same plane. The angle between the two planes is easily found. If α is the angle between the plane of action of the forces and the first principal plane of bending (the xz-plane), the equations of equilibrium become $X^{(\mathrm{iv})} = (K/I_2E)\cos\alpha$, $Y^{(\mathrm{iv})} = (K/I_1E)\sin\alpha$. The two equations differ only in the coefficient of K. Hence X and Y are proportional, and $Y = (XI_2/I_1)\tan\alpha$. The angle θ between the plane of bending and the xz-plane is given by

$$\tan\theta = (I_2/I_1)\tan\alpha. \tag{20.6}$$

For a rod of circular cross-section $I_1 = I_2$ and $\alpha = \theta$, i.e. the bending occurs in the plane of action of the forces. The same is true for a rod of any cross-section when $\alpha = 0$, i.e. when the forces act in a principal plane. The magnitude of the deflection $\zeta = \sqrt{(X^2+Y^2)}$ satisfies the equation

$$EI\zeta^{(\mathrm{iv})} = K, \qquad I = I_1I_2/\sqrt{(I_1^2\cos^2\alpha + I_2^2\sin^2\alpha)}. \tag{20.7}$$

The shearing force **F** is in the same plane as **K**, and its magnitude is

$$F = -EI\zeta'''. \tag{20.8}$$

Here I is the "effective" moment of inertia of the cross-section of the rod.

We can write down explicitly the boundary conditions on the equations of equilibrium for a slightly bent rod. If the end of the rod is clamped, we must have $X = Y = 0$ there, and also $X' = Y' = 0$, since its direction cannot change. Thus the conditions at a clamped end are

$$X = Y = 0, \qquad X' = Y' = 0. \tag{20.9}$$

The reaction force and moment at the point of support are determined from the known solution by formulae (20.3) and (20.5).

When the bending is sufficiently slight, the hinging and supporting of a point on the rod are equivalent as regards the boundary conditions. The reason is that, in the latter case, the longitudinal displacement of the rod at its point of support is of the second order of smallness compared with the

---

† An equation of the form

$$DX^{(\mathrm{iv})} - K_x = 0 \tag{20.4a}$$

also describes the bending of a thin plate in certain limiting cases. Let a rectangular plate (with sides a, b and thickness h) be fixed along its sides a (parallel to the y-axis) and bent along its sides b (parallel to the z-axis) by a load uniform in the y-direction. In the general case of arbitrary a and b, the two-dimensional equation (12.5), with the appropriate boundary conditions at the fixed and free edges, must be used to determine the bending. In the limiting case $a \gg b$, however, the deformation may be regarded as uniform in the y-direction, and then the two-dimensional equilibrium equation becomes of the form (20.4a), with the flexural rigidity replaced by $D = Eh^3a/12(1-\sigma^2)$. Equation (20.4a) is also applicable to the opposite limiting case $a \ll b$, when the plate can be regarded as a rod of length b with a narrow rectangular cross-section (a rectangle of sides a and h); in this case, however, the flexural rigidity is $D = EI_2 = Eh^3a/12$.

transverse deflection, and can therefore be neglected. The boundary conditions of zero transverse displacement and moment give

$$X = Y = 0, \qquad X'' = Y'' = 0. \tag{20.10}$$

The direction of the end of the rod and the reaction force at the point of support are obtained by solving the equations.

Finally, at a free end, the force **F** and moment **M** must be zero. According to (20.3) and (20.5), this gives the conditions

$$X'' = Y'' = 0, \qquad X''' = Y''' = 0. \tag{20.11}$$

If a concentrated force is applied at the free end, then **F** must be equal to this force, and not to zero.

It is not difficult to generalise equations (20.4) to the case of a rod of variable cross-section. For such a rod the moments of inertia I_1 and I_2 are functions of z. Formulae (20.3), which determine the moment at any cross section, are still valid. Substitution in (20.2) now gives

$$E\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(I_1\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right) = K_y, \qquad E\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(I_2\frac{\mathrm{d}^2X}{\mathrm{d}z^2}\right) = K_x, \tag{20.12}$$

in which I_1 and I_2 must be differentiated. The shearing force is

$$F_x = -E\frac{\mathrm{d}}{\mathrm{d}z}\left(I_2\frac{\mathrm{d}^2X}{\mathrm{d}z^2}\right), \qquad F_y = -E\frac{\mathrm{d}}{\mathrm{d}z}\left(I_1\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right). \tag{20.13}$$

Let us return to equations (20.1). Our neglect of the second term on the right-hand side may in some cases be illegitimate, even if the bending is slight. The cases involved are those in which a large internal stress resultant acts along the rod, i.e. F_z is very large. Such a force is usually caused by a strong tension of the rod by external stretching forces applied to its ends. We denote by T the constant lengthwise stress F_z. If the rod is strongly compressed instead of being extended, T will be negative. In expanding the vector product $\mathbf{F}\times \mathrm{d}\mathbf{t}/\mathrm{d}l$ we must now retain the terms in T, but those in F_x and F_y can again be neglected. Substituting X'', Y'', 1 for the components of the vector $\mathrm{d}\mathbf{t}/\mathrm{d}l$, we obtain the equations of equilibrium in the form

$$\begin{aligned} I_2EX^{(\mathrm{iv})} - TX'' - K_x &= 0, \\ I_1EY^{(\mathrm{iv})} - TY'' - K_y &= 0. \end{aligned} \tag{20.14}$$

The expressions (20.5) for the shearing force will now contain additional terms giving the projections of the force T (along the vector **t**) on the x and y axes:

$$F_x = -EI_2X''' + TX', \qquad F_y = -EI_1Y''' + TY'. \tag{20.15}$$

These formulae can also, of course, be obtained directly from (19.3).

In some cases a large force T can result from the bending itself, even if no stretching forces are applied. Let us consider a rod with both ends

clamped or hinged to fixed supports, so that no longitudinal displacement is possible. Then the bending of the rod must result in an extension of it, which leads to a force T in the rod. It is easy to estimate the magnitude of the deflection for which this force becomes important. The length $L + \Delta L$ of the bent rod is given by

$$L+\Delta L = \int_0^L \sqrt{(1+X'^2+Y'^2)}\,\mathrm{d}z,$$

taken along the straight line joining the points of support. For slight bending the square root can be expanded in series, and we find

$$\Delta L = \tfrac{1}{2}\int_0^L (X'^2+Y'^2)\,\mathrm{d}z.$$

The stress force in simple stretching is equal to the relative extension multiplied by YOUNG's modulus and by the area S of the cross-section of the rod. Thus the force T is

$$T = \frac{ES}{2L}\int_0^L (X'^2+Y'^2)\,\mathrm{d}z. \qquad (20.16)$$

If δ is the order of magnitude of the transverse bending, the derivatives X' and Y' are of the order of δ/L, so that the integral in (20.16) is of the order of δ^2/L, and $T \sim ES(\delta/L)^2$. The orders of magnitude of the first and second terms in (20.14) are respectively $EI\delta/L^4$ and $T\delta/L^2 \sim ES\delta^3/L^4$. The moment of inertia I is of the order of h^4, and $S \sim h^2$, where h is the thickness of the rod. Substituting, we easily find that the first and second terms in (20.14) are comparable in magnitude if $\delta \sim h$. Thus, when a rod with fixed ends is bent, the equations of equilibrium can be used in the form (20.4) only if the deflection is small in comparison with the thickness of the rod. If δ is not small compared with h (but still, of course, small compared with L), equations (20.14) must be used. The force T in these equations is not known *a priori*. It must first be regarded as a parameter in the solution, and then determined by formula (20.16) from the solution obtained; this gives the relation between T and the bending forces applied to the rod.

The opposite limiting case is that where the resistance of the rod to bending is small compared with its resistance to stretching, so that the first terms in equations (20.14) can be neglected in comparison with the second terms. Physically this case can be realized either by a very strong tension force T or by a small value of EI, which can result from a small thickness h. Rods under strong tension are called *strings*. In such cases the equations of equilibrium are

$$TX''+K_x = 0, \qquad TY''+K_y = 0. \qquad (20.17)$$

The ends of the string are fixed, in the sense that their co-ordinates are given, i.e.

$$X = Y = 0. \tag{20.18}$$

The direction of the ends cannot be decided arbitrarily, but is given by the solution of the equations.

In conclusion, we may show how the equations of equilibrium of a slightly bent rod may be obtained from the variational principle, using the expression (18.10) for the elastic energy:

$$F_{\text{rod}} = \tfrac{1}{2}E\int\{I_1 Y''^2 + I_2 X''^2\}\,dz.$$

In equilibrium the sum of this energy and the potential energy due to the external forces $\mathbf{K}$ acting on the rod must be a minimum, i.e. we must have $\delta F_{\text{rod}} - \int(K_x\delta X + K_y\delta Y)\,dz = 0$, where the second term is the work done by the external forces in an infinitesimal displacement of the rod. In varying F_{rod}, we effect a repeated integration by parts:

$$\begin{aligned}\tfrac{1}{2}\delta\int X''^2\,dz &= \int X''\delta X''\,dz \\ &= [X''\delta X'] - \int X'''\delta X'\,dz \\ &= [X''\delta X'] - [X'''\delta X] + \int X^{(\text{iv})}\delta X\,dz,\end{aligned}$$

and similarly for the integral of Y''^2. Collecting terms, we obtain

$$\int [(EI_1 Y^{(\text{iv})} - K_y)\delta Y + (EI_2 X^{(\text{iv})} - K_x)\delta X]\,dz +$$
$$+ EI_1[(Y''\delta Y' - Y'''\delta Y)] + EI_2[(X''\delta X' - X'''\delta X)] = 0.$$

The integral gives the equilibrium equations (20.4), since the variations δX and δY are arbitrary. The integrated terms give the boundary conditions on these equations; for example, at a free end the variations δX, δY, $\delta X'$, $\delta Y'$ are arbitrary, and the corresponding conditions (20.11) are obtained. Also, the coefficients of δX and δY in these terms give the expressions (20.5) for the components of the shearing force, and those of $\delta X'$ and $\delta Y'$ give the expressions (20.3) for the components of the bending moment.

Finally, the equations of equilibrium (20.14) in the presence of a tension force T can be obtained by the same method if we include in the energy a term $T\Delta L = \tfrac{1}{2}T\int(X'^2 + Y'^2)\,dz$, which is the work done by the force T over a distance ΔL equal to the extension of the rod.

PROBLEMS

PROBLEM 1. Determine the shape of a rod (of length l) bent by its own weight, for various modes of support at the ends.

SOLUTION. The required shape is given by a solution of the equation $\zeta^{(\text{iv})} = q/EI$, where q is the weight per unit length, with the appropriate boundary conditions at its ends, as shown in the text. The following shapes and maximum displacements are obtained for various modes of support at the ends of the rod. The origin is at one end of the rod in each case.

(a) Both ends clamped:

$$\zeta = qz^2(z-l)^2/24EI, \qquad \zeta(\tfrac{1}{2}l) = ql^4/384EI.$$

(b) Both ends supported:

$$\zeta = qz(z^3-2lz^2+l^3)/24EI, \qquad \zeta(\tfrac{1}{2}l) = 5ql^4/384EI.$$

(c) One end ($z = l$) clamped, the other supported:

$$\zeta = qz(2z^3-3lz^2+l^3)/48EI, \qquad \zeta(0{\cdot}42l) = 0{\cdot}0054ql^4/EI.$$

(d) One end ($z = 0$) clamped, the other free:

$$\zeta = qz^2(z^2-4lz+6l^2)/24EI, \qquad \zeta(l) = ql^4/8EI.$$

PROBLEM 2. Determine the shape of a rod bent by a force f applied to its mid-point.

SOLUTION. We have $\zeta^{(\text{iv})} = 0$ everywhere except at $z = \frac{1}{2}l$. The boundary conditions at the ends of the rod ($z = 0$ and $z = l$) are determined by the mode of support; at $z = \frac{1}{2}l$, ζ, ζ' and ζ'' must be continuous, and the discontinuity in the shearing force $F = -EI\zeta'''$ must be equal to f.

The shape of the rod (for $0 \leqslant z \leqslant \frac{1}{2}l$) and the maximum displacement are given by the following formulae:

(a) Both ends clamped:

$$\zeta = fz^2(3l-4z)/48EI, \qquad \zeta(\tfrac{1}{2}l) = fl^3/192EI.$$

(b) Both ends supported:

$$\zeta = fz(3l^2-4z^2)/48EI, \qquad \zeta(\tfrac{1}{2}l) = fl^3/48EI.$$

The rod is symmetrical about its mid-point, so that the functions $\zeta(z)$ in $\frac{1}{2}l \leqslant z \leqslant l$ are obtained simply by replacing z by $l-z$.

PROBLEM 3. The same as Problem 2, but for a rod clamped at one end ($z = 0$) and free at the other end ($z = l$), to which a force f is applied.

SOLUTION. At all points of the rod F = constant $= f$, so that $\zeta''' = -f/EI$. Using the conditions $\zeta = 0$, $\zeta' = 0$ for $z = 0$, $\zeta'' = 0$ for $z = l$, we obtain

$$\zeta = fz^2(3l-z)/6EI, \qquad \zeta(l) = fl^3/3EI.$$

PROBLEM 4. Determine the shape of a rod with fixed ends, bent by a couple at its mid-point.

SOLUTION. At all points of the rod $\zeta^{(\text{iv})} = 0$, and at $z = \frac{1}{2}l$ the moment $M = EI\zeta''$ has a discontinuity equal to the moment m of the applied couple. The results are:

(a) Both ends clamped:

$$\zeta = mz^2(l-2z)/8EIl \text{ for } 0 \leqslant z \leqslant \tfrac{1}{2}l,$$
$$\zeta = -m(l-z)^2[l-2(l-z)]/8EIl \text{ for } \tfrac{1}{2}l \leqslant z \leqslant l.$$

(b) Both ends hinged:

$$\zeta = mz(l^2-4z^2)/24EIl \text{ for } 0 \leqslant z \leqslant \tfrac{1}{2}l,$$
$$\zeta = -m(l-z)[l^2-4(l-z)^2]/24EIl \text{ for } \tfrac{1}{2}l \leqslant z \leqslant l.$$

The rod is bent in opposite directions on the two sides of $z = \frac{1}{2}l$.

PROBLEM 5. The same as Problem 4, but for the case where one end is clamped and the other end free, the couple being applied at the latter end.

SOLUTION. At all points of the rod $M = EI\zeta'' = m$, and at $z = 0$ we have $\zeta = 0$, $\zeta' = 0$. The shape is given by $\zeta = mz^2/2EI$.

PROBLEM 6. Determine the shape of a circular rod with hinged ends stretched by a force T and bent by a force f applied at its mid-point.

SOLUTION. On the segment $0 \leqslant z \leqslant \frac{1}{2}l$ the shearing force is $\frac{1}{2}f$, so that (20.15) gives the equation

$$\zeta''' - T\zeta'/EI = -f/2EI.$$

The boundary conditions are $\zeta = \zeta'' = 0$ for $z = 0$ and l; $\zeta' = 0$ for $z = \frac{1}{2}l$ (since ζ' is continuous). The shape of the rod (in the segment $0 \leqslant z \leqslant \frac{1}{2}l$) is given by

$$\zeta = \frac{f}{2T}\left(z - \frac{\sinh kz}{k\cosh\frac{1}{2}kl}\right), \qquad k = \sqrt{(T/EI)}.$$

For small k this gives the result obtained in Problem 2 (b). For large k it becomes $\zeta = fz/2T$, i.e., in accordance with equations (20.17), a flexible wire under a force f takes the form of two straight pieces intersecting at $z = \frac{1}{2}l$.

If the force T is due to the stretching of the rod by the transverse force, it must be determined by formula (20.16). Substituting the above result, we obtain the equation

$$\frac{1}{k^6}\left[\frac{3}{2}+\frac{1}{2}\tanh^2\frac{1}{2}kl - \frac{3}{kl}\tanh\frac{1}{2}kl\right] = \frac{8E^2I^3}{f^2S},$$

which determines T as an implicit function of f.

PROBLEM 7. A circular rod of infinite length lies in an elastic substance, i.e. when it is bent a force $K = -\alpha\zeta$ proportional to the deflection acts on it. Determine the shape of the rod when a concentrated force f acts on it.

SOLUTION. We take the origin at the point where the force f is applied. The equation $EI\zeta^{(iv)} = -\alpha\zeta$ holds everywhere except at $z = 0$. The solution must satisfy the condition $\zeta = 0$ at $z = \pm\infty$, and at $z = 0$ ζ' and ζ'' must be continuous; the difference between the shearing forces $F = -EI\zeta'''$ for $z \to 0+$ and $z \to 0-$ must be f. The required solution is

$$\zeta = \frac{f}{8\beta^3EI}e^{-\beta|z|}[\cos\beta|z| + \sin\beta|z|], \qquad \beta = \left(\frac{\alpha}{4EI}\right)^{1/4}.$$

PROBLEM 8. Derive the equation of equilibrium for a slightly bent thin circular rod which, in its undeformed state, is an arc of a circle and is bent in its plane by radial forces.

SOLUTION. Taking the origin of polar co-ordinates r, ϕ at the centre of the circle, we write the equation of the deformed rod as $r = a+\zeta(\phi)$, where a is the radius of the arc and ζ a small radial displacement. Using the expression for the radius of curvature in polar co-ordinates, we find as far as the first order in ζ

$$\frac{1}{R} = \frac{r^2 - rr'' + 2r'^2}{(r^2+r'^2)^{3/2}} \approx \frac{1}{a} - \frac{\zeta+\zeta''}{a^2},$$

where the prime denotes differentiation with respect to ϕ. According to (18.11), the elastic bending energy is

$$F_{\text{rod}} = \tfrac{1}{2}EI\int_0^{\phi_0}\left(\frac{1}{R} - \frac{1}{a}\right)^2 a\,\mathrm{d}\phi = \frac{EI}{2a^3}\int_0^{\phi_0}(\zeta+\zeta'')^2\,\mathrm{d}\phi,$$

ϕ_0 being the angle subtended by the arc at its centre. The equation of equilibrium is obtained from the variational principle

$$\delta F_{\text{rod}} - \int_0^{\phi_0} \delta\zeta K_r a \, \mathrm{d}\phi = 0,$$

where K_r is the external radial force per unit length, with the auxiliary condition

$$\int_0^{\phi_0} \zeta \, \mathrm{d}\phi = 0,$$

which is, in this approximation, the statement of the fact that the total length of the rod is unchanged, i.e. it undergoes no general extension. Using LAGRANGE's method, we put

$$\delta F_{\text{rod}} - \int_0^{\phi_0} aK_r\delta\zeta \, \mathrm{d}\phi + a\alpha \int_0^{\phi_0} \delta\zeta \, \mathrm{d}\phi = 0,$$

where α is a constant. Varying the integrand in F_{rod} and integrating the $\delta\zeta''$ term twice by parts, we obtain

$$\int \left\{ \frac{EI}{a^3}(\zeta + 2\zeta'' + \zeta^{(\text{iv})}) - aK_r + \alpha a \right\} \delta\zeta \, \mathrm{d}\phi +$$

$$+ \frac{EI}{a^3}[(\zeta + \zeta'')\delta\zeta'] - \frac{EI}{a^3}[(\zeta' + \zeta''')\delta\zeta] = 0.$$

Hence we find the equation of equilibrium†

$$EI(\zeta^{(\text{iv})} + 2\zeta'' + \zeta)/a^4 - K_r + \alpha = 0, \qquad (1)$$

the shearing force $F = -EI(\zeta' + \zeta''')/a^3$, and the bending moment $M = EI(\zeta + \zeta'')/a^2$; cf. the end of §20. The constant α is determined from the condition that the rod as a whole is not stretched.

PROBLEM 9. Determine the deformation of a circular ring bent by two forces f applied along a diameter (Fig. 18).

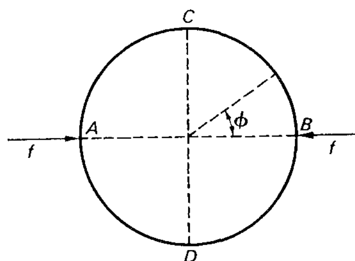


FIG. 18

† In the absence of external forces, $K_r = 0$ and $\alpha = 0$; the non-zero solutions of the resulting homogeneous equation correspond to a simple rotation or translation of the whole rod.

SOLUTION. Integrating equation (1), Problem 8, along the circumference of the ring, we have $2\pi\alpha a = \int K_r a \, d\phi - 2f$. We have equation (1) with $K_r = 0$ everywhere except at $\phi = 0$ and $\phi = \pi$:

$$\zeta^{(\mathrm{iv})} + 2\zeta'' + \zeta + fa^3/\pi EI = 0.$$

The required deformation of the ring is symmetrical about the diameters AB and CD, and so we must have $\zeta' = 0$ at A, B, C and D. The difference in the shearing forces for $\phi \to 0$ must be f. The solution of the equation of equilibrium which satisfies these conditions is

$$\zeta = \frac{fa^3}{EI}\left(\frac{1}{\pi} + \frac{1}{4}\phi \cos\phi - \frac{1}{8}\pi \cos\phi - \frac{1}{4}\sin\phi\right), \qquad 0 \leqslant \phi \leqslant \pi.$$

In particular, the points A and B approach through a distance

$$|\zeta(0) + \zeta(\pi)| = \frac{fa^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right).$$

§21. The stability of elastic systems

The behaviour of a rod subject to longitudinal compressing forces is the simplest example of the important phenomenon of *elastic instability*, first discovered by L. EULER.

In the absence of transverse bending forces K_x, K_y, the equations of equilibrium (20.14) for a compressed rod have the evident solution $X = Y = 0$, which corresponds to the rod's remaining straight under a longitudinal force $|T|$. This solution, however, gives a stable equilibrium of the rod only if the compressing force $|T|$ is less than a certain critical value T_{cr}. For $|T| < T_{\mathrm{cr}}$, the straight rod is stable with respect to any small perturbation. In other words, if the rod is slightly bent by some small force, it will tend to return to its original position when that force ceases to act.

If, on the other hand, $|T| > T_{\mathrm{cr}}$, the straight rod is in unstable equilibrium. An infinitesimal bending suffices to destroy the equilibrium, and a large bending of the rod results. It is clear that, if this is so, the compressed rod cannot actually remain straight.

The behaviour of the rod after it ceases to be stable must satisfy the equations for bending with large deflections. The value T_{cr} of the critical load, however, can be obtained from the equations for small deflections. For $|T| = T_{\mathrm{cr}}$, the straight rod is in neutral equilibrium. This means that, besides the solution $X = Y = 0$, there must also be states where the rod is slightly bent but still in equilibrium. Hence the critical value of T_{cr} is the value of $|T|$ for which the equations

$$EI_2 X^{(\mathrm{iv})} + |T| X'' = 0, \qquad EI_1 Y^{(\mathrm{iv})} + |T| Y'' = 0 \qquad (21.1)$$

have a non-zero solution. This solution gives also the nature of the deformation of the rod immediately after it ceases to be stable.

The following Problems give some typical cases of the loss of stability in various elastic systems.

PROBLEMS

PROBLEM 1. Determine the critical compression force for a rod with hinged ends.

SOLUTION. Since we are seeking the smallest value of $|T|$ for which equations (21.1) have a non-zero solution, it is sufficient to consider only the equation which contains the smaller of I_1 and I_2. Let $I_2 < I_1$. Then we seek a solution of the equation $EI_2X^{(\text{iv})}+|T|X'' = 0$ in the form $X = A+Bz+C\sin kz+D\cos kz$, where $k = \sqrt{(|T|/EI_2)}$. The non-zero solution which satisfies the conditions $X = X'' = 0$ for $z = 0$ and $z = l$ is $X = C\sin kz$, with $\sin kl = 0$. Hence we find the required critical force to be $T_{\text{cr}} = \pi^2EI_2/l^2$. On ceasing to be stable, the rod takes the form shown in Fig. 19a.

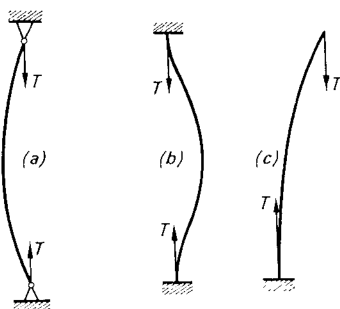


FIG. 19

PROBLEM 2. The same as Problem 1, but for a rod with clamped ends (Fig. 19b).

SOLUTION. $T_{\text{cr}} = 4\pi^2EI_2/l^2$.

PROBLEM 3. The same as Problem 1, but for a rod with one end clamped and the other free (Fig. 19c).

SOLUTION. $T_{\text{cr}} = \pi^2EI_2/4l^2$.

PROBLEM 4. Determine the critical compression force for a circular rod with hinged ends in an elastic medium (see §20, Problem 7).

SOLUTION. The equations (21.1) must now be replaced by $EIX^{(\text{iv})}+|T|X''+\alpha X = 0$. A similar treatment gives the solution $X = A\sin n\pi z/l$,

$$T_{\text{cr}} = \frac{\pi^2EI}{l^2}\left(n^2+\frac{\alpha l^4}{n^2\pi^4EI}\right),$$

where n is the integer for which T_{cr} is least. When α is large, $n > 1$, i.e. the rod exhibits several undulations as soon as it ceases to be stable.

PROBLEM 5. A circular rod is subjected to torsion, its ends being clamped. Determine the critical torsion beyond which the straight rod becomes unstable.

SOLUTION. The critical value of the torsion angle is determined by the appearance of non-zero solutions of the equations for slight bending of a twisted rod. To derive these equations, we substitute the expression (19.7) $\mathbf{M} = EI\mathbf{t}\times \mathrm{d}\mathbf{t}/\mathrm{d}l+C\tau\mathbf{t}$, where τ is the constant torsion angle, in equation (19.3). This gives

$$EI\mathbf{t}\times\frac{\mathrm{d}^2\mathbf{t}}{\mathrm{d}l^2}+C\tau\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}l} - \mathbf{F}\times\mathbf{t} = 0.$$

We differentiate; since the bending is not large, **t** may be regarded as a constant vector $\mathbf{t}_0$ along the axis of the rod (the z-axis) in differentiating the first and third terms. Since also $d\mathbf{F}/dl = 0$ (there being no external forces except at the ends of the rod), we obtain

$$EI\mathbf{t}_0 \times \frac{d^3\mathbf{t}}{dl^3} + C\tau\frac{d^2\mathbf{t}}{dl^2} = 0,$$

or, in components,

$$Y^{(\mathrm{iv})} - \kappa X''' = 0,$$
$$X^{(\mathrm{iv})} + \kappa Y''' = 0,$$

where $\kappa = C\tau/EI$. Taking as the unknown function $\xi = X+iY$, we obtain $\xi^{(\mathrm{iv})} - i\kappa\xi''' = 0$. We seek a solution which satisfies the conditions $\xi = 0$, $\xi' = 0$ for $z = 0$ and $z = l$, in the form $\xi = a(1+i\kappa z - e^{i\kappa z}) + bz^2$, and obtain as the compatibility condition of the equations for a and b the relation $e^{i\kappa l} = (2+i\kappa l)/(2-i\kappa l)$, whence $\frac{1}{2}\kappa l = \tan\frac{1}{2}\kappa l$. The smallest root of this equation is $\frac{1}{2}\kappa l = 4 \cdot 49$, so that $\tau_{\mathrm{cr}} = 8 \cdot 98EI/Cl$.

PROBLEM 6. The same as Problem 5, but for a rod with hinged ends.

SOLUTION. In this case we have $\xi = a(1-e^{i\kappa z}-\frac{1}{2}\kappa^2 z^2)+bz$, where κ is given by

$$e^{i\kappa l} = 1, \text{ i.e. } \kappa l = 2\pi.$$

Hence the required critical torsion angle is $\tau_{\mathrm{cr}} = 2\pi EI/Cl$.

PROBLEM 7. Determine the limit of stability of a vertical rod under its own weight, the ower end being clamped.

SOLUTION. If the longitudinal stress $F_z \equiv T$ varies along the rod, $dF_z/dl \neq 0$ in the first term of (20.1), and equations (20.14) are replaced by

$$I_2 EX^{(\mathrm{iv})} - (TX')' - K_x = 0,$$
$$I_1 EY^{(\mathrm{iv})} - (TY')' - K_y = 0.$$

In the case considered, there are no transverse bending forces anywhere in the rod, and $T = -q(l-z)$, where q is the weight of the rod per unit length and z is measured from the lower end. Assuming that $I_2 < I_1$, we consider the equation

$$I_2 EX''' = TX' = -q(l-z)X';$$

for $z = l$, $X''' = 0$ automatically. The general integral of this equation for the function $u = X'$ is

$$u = \eta^{\frac{1}{3}}[aJ_{-\frac{1}{3}}(\eta)+bJ_{\frac{1}{3}}(\eta)],$$

where

$$\eta = \tfrac{2}{3}\sqrt{[q(l-z)^3/EI_2]}.$$

The boundary conditions $X' = 0$ for $z = 0$ and $X'' = 0$ for $z = l$ give for the function $u(\eta)$ the conditions $u = 0$ for $\eta = \eta_0 \equiv \frac{2}{3}\sqrt{(ql^3/EI_2)}$, $u'\eta^{1/3} = 0$ for $\eta = 0$. In order to satisfy these conditions we must put $b = 0$ and $J_{-\frac{1}{3}}(\eta_0) = 0$. The smallest root of this equation is $\eta_0 = 1 \cdot 87$, and so the critical length is $l_{\mathrm{cr}} = 1 \cdot 98(EI_2/q)^{1/3}$.

PROBLEM 8. A rod has an elongated cross-section, so that $I_2 \gg I_1$. One end is clamped and a force f is applied to the other end, which is free, so as to bend it in the principal xz-plane (in which the flexural rigidity is EI_2). Determine the critical force f_{cr} at which the rod bent in a plane becomes unstable and the rod is bent sideways (in the yz-plane), at the same time undergoing torsion.

SOLUTION. Since the rigidity EI_2 is large compared with EI_1 (and with the torsional rigidity C),† the instability as regards sideways bending occurs while the deflection in the xz-plane is still small. To determine the point where instability sets in, we must form the equations for slight sideways bending of the rod, retaining the terms proportional to the products of the force f in the xz-plane and the small displacements. Since there is a concentrated force only at the free end of the rod, we have $\mathbf{F} = \mathbf{f}$ at all points, and at the free end ($z = l$) the moment $\mathbf{M} = 0$; from formula (19.6) we find the components of the moment relative to a fixed system of co-ordinates x, y, z: $M_x = 0$, $M_y = (l-z)f$, $M_z = (Y-Y_0)f$, where $Y_0 = Y(l)$. Taking the components along co-ordinate axes ξ, η, ζ fixed at each point to the rod, we obtain as far as the first-order terms in the displacements $M_\xi = \phi(l-z)f$, $M_\eta = (l-z)f$, $M_\zeta = (l-z)f\,\mathrm{d}Y/\mathrm{d}z + f(Y-Y_0)$, where ϕ is the total angle of rotation of a cross-section of the rod under torsion; the torsion angle $\tau = \mathrm{d}\phi/\mathrm{d}z$ is not constant along the rod. According to (18.6) and (18.9), however, we have for a small deflection

$$M_\xi = -EI_1Y'', \qquad M_\eta = EI_2X'', \qquad M_\zeta = C\phi';$$

comparing, we obtain the equations of equilibrium

$$EI_2X'' = (l-z)f, \qquad EI_1Y'' = -\phi(l-z)f,$$
$$C\phi' = (l-z)fY' + (Y-Y_0)f.$$

The first of these equations gives the main bending of the rod, in the xz-plane; we require the value of f for which non-zero solutions of the second and third equations appear. Eliminating Y, we find

$$\phi'' + k^2(l-z)^2\phi = 0, \qquad k^2 = f^2/EI_1C.$$

The general integral of this equation is

$$\phi = a\sqrt{(l-z)}J_{\frac{1}{4}}[\tfrac{1}{2}k(l-z)^2] + b\sqrt{(l-z)}J_{-\frac{1}{4}}[\tfrac{1}{2}k(l-z)^2].$$

At the clamped end ($z = 0$) we must have $\phi = 0$, and at the free end the twisting moment $C\phi' = 0$. From the second condition we have $a = 0$, and then the first gives $J_{-\frac{1}{4}}(\frac{1}{2}kl^2) = 0$. The smallest root of this equation is $\frac{1}{2}kl^2 = 2{\cdot}006$, whence $f_{\mathrm{cr}} = 4{\cdot}01\sqrt{(EI_1C)}/l^2$.

† For example, for a narrow rectangular cross-section of sides b and h ($b \gg h$), we have $EI_1 = bh^3E/12$, $EI_2 = b^3hE/12$, $C = bh^3\mu/3$.

CHAPTER III

ELASTIC WAVES

§22. Elastic waves in an isotropic medium

IF motion occurs in a deformed body, its temperature is not in general constant, but varies in both time and space. This considerably complicates the exact equations of motion in the general case of arbitrary motions.

Usually, however, matters are simplified in that the transfer of heat from one part of the body to another (by simple thermal conduction) occurs very slowly. If the heat exchange during times of the order of the period of oscillatory motions in the body is negligible, we can regard any part of the body as thermally insulated, i.e. the motion is adiabatic. In adiabatic deformations, however, σ_{ik} is given in terms of u_{ik} by the usual formulae, the only difference being that the ordinary (isothermal) values of E and σ must be replaced by their adiabatic values (see §6). We shall assume in what follows that this condition is fulfilled, and accordingly E and σ in this chapter will be understood to have their adiabatic values.

In order to obtain the equations of motion for an elastic medium, we must equate the internal stress force $\partial\sigma_{ik}/\partial x_k$ to the product of the acceleration $\ddot{u}_i$ and the mass per unit volume of the body, i.e. its density ρ:

$$\rho\ddot{u}_i = \partial\sigma_{ik}/\partial x_k. \tag{22.1}$$

This is the general equation of motion.

In particular, the equations of motion for an isotropic elastic medium can be written down at once by analogy with the equation of equilibrium (7.2). We have

$$\rho\ddot{\mathbf{u}} = \frac{E}{2(1+\sigma)}\triangle\mathbf{u} + \frac{E}{2(1+\sigma)(1-2\sigma)}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u}. \tag{22.2}$$

Since all deformations are supposed small, the motions considered in the theory of elasticity are small *elastic oscillations* or *elastic waves*. We shall begin by discussing a plane elastic wave in an infinite isotropic medium, i.e. a wave in which the deformation $\mathbf{u}$ is a function only of one co-ordinate (x, say) and of the time. All derivatives with respect to y and z in equations (22.2) are then zero, and we obtain for the components of the vector $\mathbf{u}$ the equations

$$\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c_l^2}\frac{\partial^2 u_x}{\partial t^2} = 0, \qquad \frac{\partial^2 u_y}{\partial x^2} - \frac{1}{c_t^2}\frac{\partial^2 u_y}{\partial t^2} = 0 \tag{22.3}$$

(the equation for u_z is the same as that for u_y); here†

$$c_l = \sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}, \qquad c_t = \sqrt{\frac{E}{2\rho(1+\sigma)}}. \tag{22.4}$$

Equations (22.3) are ordinary wave equations in one dimension, and the quantities c_l and c_t which appear in them are the velocities of propagation of the wave. We see that the velocity of propagation for the component u_x is different from that for u_y and u_z.

Thus an elastic wave is essentially two waves propagated independently. In one (u_x) the displacement is in the direction of propagation; this is called the *longitudinal wave,* and is propagated with velocity c_l. In the other wave (u_y, u_z) the displacement is in a plane perpendicular to the direction of propagation; this is called the *transverse wave,* and is propagated with velocity c_t. It is seen from (22.4) that the velocity of longitudinal waves is always greater than that of transverse waves: we always have‡

$$c_l > \sqrt{(4/3)}c_t. \tag{22.5}$$

The velocities c_l and c_t are often called the *longitudinal and transverse velocities of sound.*

We know that the volume change in a deformation is given by the sum of the diagonal terms in the strain tensor, i.e. by $u_{ii} \equiv \operatorname{div} \mathbf{u}$. In the transverse wave there is no component u_x, and, since the other components do not depend on y or z, $\operatorname{div} \mathbf{u} = 0$ for such a wave. Thus transverse waves do not involve any change in volume of the parts of the body. For longitudinal waves, however, $\operatorname{div} \mathbf{u} \neq 0$, and these waves involve compressions and expansions in the body.

The separation of the wave into two parts propagated independently with different velocities can also be effected in the general case of an arbitrary (not plane) elastic wave in an infinite medium. We rewrite equation (22.2) in terms of the velocities c_l and c_t:

$$\ddot{\mathbf{u}} = c_t^2 \triangle \mathbf{u} + (c_l^2 - c_t^2)\, \mathbf{grad}\, \operatorname{div} \mathbf{u}. \tag{22.6}$$

We then represent the vector $\mathbf{u}$ as the sum of two parts:

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \tag{22.7}$$

of which one satisfies

$$\operatorname{div} \mathbf{u}_t = 0 \tag{22.8}$$

and the other satisfies

$$\mathbf{curl}\, \mathbf{u}_l = 0. \tag{22.9}$$

We know from vector analysis that this representation (i.e. the expression of

† We may give also expressions for c_l and c_t in terms of the moduli of compression and rigidity and the Lamé coefficients: $c_l = \sqrt{\{(3K + 4\mu)/3\rho\}} = \sqrt{\{(\lambda + 2\mu)/\rho\}}$, $c_t = \sqrt{(\mu/\rho)}$.

‡ Since σ actually varies only between 0 and $\frac{1}{2}$ (see the second footnote to §5), we always have $c_l > \sqrt{2}c_t$.

a vector as the sum of the curl of a vector and the gradient of a scalar) is always possible.

Substituting $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$ in (22.6), we obtain

$$\ddot{\mathbf{u}}_l + \ddot{\mathbf{u}}_t = c_t^2 \triangle (\mathbf{u}_l + \mathbf{u}_t) + (c_l^2 - c_t^2)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u}_l. \tag{22.10}$$

We take the divergence of both sides. Since $\mathrm{div}\,\mathbf{u}_t = 0$, the result is

$$\mathrm{div}\,\ddot{\mathbf{u}}_l = c_t^2 \triangle\, \mathrm{div}\,\mathbf{u}_l + (c_l^2 - c_t^2) \triangle\, \mathrm{div}\,\mathbf{u}_l,$$

or $\mathrm{div}(\ddot{\mathbf{u}}_l - c_l^2 \triangle \mathbf{u}_l) = 0$. The curl of the expression in parentheses is also zero, by (22.9). If the curl and divergence of a vector both vanish in all space, that vector must be zero identically. Thus

$$\frac{\partial^2 \mathbf{u}_l}{\partial t^2} - c_l^2 \triangle \mathbf{u}_l = 0. \tag{22.11}$$

Similarly, taking the curl of equation (22.10) we have, since the curls of $\mathbf{u}_l$ and of any gradient are zero, $\mathbf{curl}\,(\ddot{\mathbf{u}}_t - c_t^2 \triangle \mathbf{u}_t) = 0$. Since the divergence of the expression in parentheses is also zero, we obtain an equation of the same form as (22.11):

$$\frac{\partial^2 \mathbf{u}_t}{\partial t^2} - c_t^2 \triangle \mathbf{u}_t = 0. \tag{22.12}$$

Equations (22.11) and (22.12) are ordinary wave equations in three dimensions. Each of them represents the propagation of an elastic wave, with velocity c_l and c_t respectively. One wave ($\mathbf{u}_t$) does not involve a change in volume (since $\mathrm{div}\,\mathbf{u}_t = 0$), while the other ($\mathbf{u}_l$) is accompanied by volume compressions and expansions.

In a monochromatic elastic wave, the displacement vector is

$$\mathbf{u} = \mathrm{re}\{\mathbf{u}_0(\mathbf{r}) e^{-i\omega t}\}, \tag{22.13}$$

where $\mathbf{u}_0$ is a function of the co-ordinates which satisfies the equation

$$c_t^2 \triangle \mathbf{u}_0 + (c_l^2 - c_t^2)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u}_0 + \omega^2 \mathbf{u}_0 = 0, \tag{22.14}$$

obtained by substituting (22.13) in (22.6). The longitudinal and transverse parts of a monochromatic wave satisfy the equations

$$\triangle \mathbf{u}_l + k_l^2 \mathbf{u}_l = 0, \qquad \triangle \mathbf{u}_t + k_t^2 \mathbf{u}_t = 0, \tag{22.15}$$

where $k_l = \omega/c_l$, $k_t = \omega/c_t$ are the wave numbers of the longitudinal and transverse waves.

Finally, let us consider the reflection and refraction of a plane monochromatic elastic wave at the boundary between two different elastic media. It must be borne in mind that the nature of the wave is in general changed when it is reflected or refracted. If a purely transverse or purely longitudinal wave is incident on a surface of separation, the result is a mixed wave containing both transverse and longitudinal parts. The nature of the wave

remains unchanged (as we see from symmetry) only when it is incident normally on the surface of separation, or when a transverse wave whose oscillations are parallel to the plane of separation is incident (at any angle).

The relations giving the directions of the reflected and refracted waves can be obtained immediately from the constancy of the frequency and of the tangential components of the wave vector.† Let θ and θ' be the angles of incidence and reflection (or refraction) and c, c' the velocities of the two waves. Then

$$\frac{\sin\theta}{\sin\theta'} = \frac{c}{c'}. \tag{22.16}$$

For example, let the incident wave be transverse. Then $c = c_{t1}$ is the velocity of transverse waves in medium 1. For the transverse reflected wave we have $c' = c_{t1}$ also, so that (22.16) gives $\theta = \theta'$, i.e. the angle of incidence is equal to the angle of reflection. For the longitudinal reflected wave, however, $c' = c_{l1}$, and so

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{l1}}.$$

For the transverse part of the refracted wave $c' = c_{t2}$, and for a transverse incident wave

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{t2}}.$$

Similarly, for the longitudinal refracted wave

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{l2}}.$$

PROBLEMS

PROBLEM 1. Determine the reflection coefficient for a longitudinal monochromatic wave incident at any angle on the surface of a body (with a vacuum outside).‡

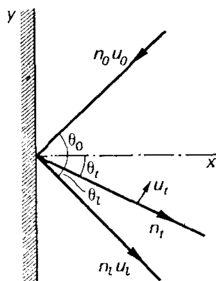


FIG. 20

SOLUTION. When the wave is reflected, there are in general both longitudinal and transverse reflected waves. It is clear from symmetry that the displacement vector in the transverse reflected wave lies in the plane of incidence (Fig. 20, where $\mathbf{n}_0$, $\mathbf{n}_l$ and $\mathbf{n}_t$ are unit

† See *Fluid Mechanics*, §65. The arguments given there are applicable in their entirety.

‡ The more general case of the reflection of sound waves from a solid liquid interface, and the similar problem of the reflection of a wave incident from a liquid on to a solid, are discussed by L. M. BREKHOVSKIKH, *Waves in Layered Media*, §4, Academic Press, New York 1960.

vectors in the direction of propagation of the incident, longitudinal reflected and transverse reflected waves, and $\mathbf{u}_0$, $\mathbf{u}_l$, $\mathbf{u}_t$ the corresponding displacement vectors). The total displacement in the body is given by the sum (omitting the common factor $e^{-i\omega t}$ for brevity)

$$\mathbf{u} = A_0\mathbf{n}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}} + A_l\mathbf{n}_l e^{i\mathbf{k}_l\cdot\mathbf{r}} + A_t\mathbf{a}\times\mathbf{n}_t e^{i\mathbf{k}_t\cdot\mathbf{r}},$$

where $\mathbf{a}$ is a unit vector perpendicular to the plane of incidence. The magnitudes of the wave vectors are $k_0 = k_l = \omega/c_l$, $k_t = \omega/c_t$, and the angles of incidence θ_0 and of reflection θ_l, θ_t are related by $\theta_l = \theta_0$, $\sin\theta_t = (c_t/c_l)\sin\theta_0$. For the components of the strain tensor at the boundary we obtain

$$u_{xx} = ik_0(A_0 + A_l)\cos^2\theta_0 + iA_tk_t\cos\theta_t\sin\theta_t, \quad u_{ll} = ik_0(A_0 + A_l),$$
$$u_{xy} = ik_0(A_0 - A_l)\sin\theta_0\cos\theta_0 + \tfrac{1}{2}iA_tk_t(\cos^2\theta_t - \sin^2\theta_t),$$

again omitting the common exponential factor. The components of the stress tensor can be calculated from the general formula (5.11), which can here be conveniently written

$$\sigma_{ik} = 2\rho c_t^2 u_{ik} + \rho(c_l^2 - 2c_t^2)u_{ll}\delta_{ik}.$$

The boundary conditions at the free surface of the medium are $\sigma_{ik}n_k = 0$, whence

$$\sigma_{xx} = \sigma_{yx} = 0,$$

giving two equations which express A_l and A_t in terms of A_0. The result is

$$A_l = A_0\frac{c_t^2\sin 2\theta_t\sin 2\theta_0 - c_l^2\cos^2 2\theta_t}{c_t^2\sin 2\theta_t\sin 2\theta_0 + c_l^2\cos^2 2\theta_t},$$

$$A_t = -A_0\frac{2c_lc_t\sin 2\theta_0\cos 2\theta_t}{c_t^2\sin 2\theta_t\sin 2\theta_0 + c_l^2\cos^2 2\theta_t}.$$

For $\theta_0 = 0$ we have $A_l = -A_0$, $A_t = 0$, i.e. the wave is reflected as a purely longitudinal wave. The ratio of the energy flux density components normal to the surface in the reflected and incident longitudinal waves is $R_l = |A_l/A_0|^2$. The corresponding ratio for the reflected transverse wave is

$$R_t = \frac{c_t\cos\theta_t}{c_l\cos\theta_0}\left|\frac{A_t}{A_0}\right|^2.$$

The sum of R_l and R_t is, of course, 1.

Problem 2. The same as Problem 1, but for a transverse incident wave (with the oscillations in the plane of incidence).†

Solution. The wave is reflected as a transverse and a longitudinal wave, with $\theta_t = \theta_0$, $c_t\sin\theta_l = c_l\sin\theta_0$. The total displacement vector is

$$\mathbf{u} = \mathbf{a}\times\mathbf{n}_0A_0e^{i\mathbf{k}_0\cdot\mathbf{r}} + \mathbf{n}_lA_le^{i\mathbf{k}_l\cdot\mathbf{r}} + \mathbf{a}\times\mathbf{n}_tA_te^{i\mathbf{k}_t\cdot\mathbf{r}}.$$

The expressions for the amplitudes of the reflected waves are

$$\frac{A_t}{A_0} = \frac{c_t^2\sin 2\theta_l\sin 2\theta_0 - c_l^2\cos^2 2\theta_0}{c_t^2\sin 2\theta_l\sin 2\theta_0 + c_l^2\cos^2 2\theta_0},$$

$$\frac{A_l}{A_0} = \frac{2c_lc_t\sin 2\theta_0\cos 2\theta_0}{c_t^2\sin 2\theta_l\sin 2\theta_0 + c_l^2\cos^2 2\theta_0}.$$

† If the oscillations are perpendicular to the plane of incidence, the wave is entirely reflected as a wave of the same kind, and so $R_t = 1$.

PROBLEM 3. Determine the characteristic frequencies of radial vibrations of an elastic phere of radius R.

SOLUTION. We take spherical polar co-ordinates, with the origin at the centre of the sphere. For radial vibrations, $\mathbf{u}$ is along the radius, and is a function of r and t only. Hence $\mathbf{curl}\,\mathbf{u} = 0$. We define the displacement "potential" ϕ by $u_r = u = \partial\phi/\partial r$. The equation of motion, expressed in terms of ϕ, is just the wave equation $c_l^2\triangle\phi = \ddot{\phi}$, or, for oscillations periodic in time ($\sim e^{-i\omega t}$),

$$\triangle\phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -k^2\phi,\ k = \omega/c_l. \tag{1}$$

The solution which is finite at the origin is $\phi = (A/r)\sin kr$ (the time factor is omitted). The radial stress is

$$\sigma_{rr} = \rho\left\{(c_l^2-2c_t^2)u_{ii}+2c_t^2u_{rr}\right\}$$

$$= \rho\left\{(c_l^2-2c_t^2)\triangle\phi+2c_t^2\phi''\right\}$$

or, using (1),

$$\sigma_{rr}/\rho = -\omega^2\phi-4c_t^2\phi'/r. \tag{2}$$

The boundary condition $\sigma_{rr}(R) = 0$ leads to the equation

$$\frac{\tan kR}{kR} = \frac{1}{1-(kRc_l/2c_t)^2}, \tag{3}$$

whose roots determine the characteristic frequencies $\omega = kc_l$ of the vibrations.

PROBLEM 4. Determine the frequency of radial vibrations of a spherical cavity in an infinite elastic medium for which $c_l \gg c_t$ (M. A. ISAKOVICH 1949).

SOLUTION. In an infinite medium, radial oscillations of the cavity are accompanied by the emission of longitudinal sound waves, leading to loss of energy and hence to damping of the oscillations. When $c_l \gg c_t$ (i.e. $K \gg \mu$), this emission is weak, and we can speak of the characteristic frequencies of oscillations with a small coefficient of damping.

We seek a solution of equation (1), Problem 3, in the form of an outgoing spherical wave $\phi = Ae^{ikr}/r$, $k = \omega/c_l$ and, using (2), obtain from the boundary condition $\sigma_{rr}(R) = 0$ the result $(kRc_l/c_t)^2 = 4(1-ikR)$. Hence, when $c_l \gg c_t$,

$$\omega = \frac{2c_t}{R}\left(1-i\frac{c_t}{c_l}\right).$$

The real part of ω gives the characteristic frequency of oscillation; the imaginary part gives the damping coefficient. In an incompressible medium ($c_l\to\infty$) there would of course be no damping. These vibrations are specifically due to the shear resistance of the medium ($\mu \neq 0$). It should be noticed that they have $kR = 2c_t/c_l \ll 1$, i.e. the corresponding wavelength is large compared with R; it is interesting to compare this with the result for vibrations of an elastic sphere, where with $c_l \gg c_t$ the first characteristic frequency is given by (3): $kR = \pi$.

§23. Elastic waves in crystals

The propagation of elastic waves in anisotropic media, i.e. in crystals, is more complicated than for the case of isotropic media. To investigate such waves, we must return to the general equations of motion $\rho\ddot{u}_i = \partial\sigma_{ik}/\partial x_k$ and use for σ_{ik} the general expression (10.3) $\sigma_{ik} = \lambda_{iklm}u_{lm}$. According to

what was said at the beginning of §22, λ_{iklm} always denotes the adiabatic moduli of elasticity.

Substituting for σ_{ik} in the equations of motion, we obtain

$$\rho\ddot{u}_i = \lambda_{iklm}\frac{\partial u_{lm}}{\partial x_k} = \tfrac{1}{2}\lambda_{iklm}\frac{\partial}{\partial x_k}\left(\frac{\partial u_l}{\partial x_m}+\frac{\partial u_m}{\partial x_l}\right)$$

$$= \tfrac{1}{2}\lambda_{iklm}\frac{\partial^2 u_l}{\partial x_k \partial x_m}+\tfrac{1}{2}\lambda_{iklm}\frac{\partial^2 u_m}{\partial x_k \partial x_l}.$$

Since the tensor λ_{iklm} is symmetrical with respect to the suffixes l and m we can interchange these in the first term, which then becomes identical with the second term. Thus the equations of motion are

$$\rho\ddot{u}_i = \lambda_{iklm}\frac{\partial^2 u_m}{\partial x_k \partial x_l}. \tag{23.1}$$

Let us consider a monochromatic elastic wave in a crystal. We can seek a solution of the equations of motion in the form $u_i = u_{0i}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, where the u_{0i} are constants, the relation between the wave vector $\mathbf{k}$ and the frequency ω being such that this function actually satisfies equation (23.1). Differentiation of u_i with respect to time results in multiplication by $-i\omega$, and differentiation with respect to x_k leads to multiplication by ik_k. Hence the above substitution converts equation (23.1) into $\rho\omega^2 u_i = \lambda_{iklm}k_k k_l u_m$. Putting $u_i = \delta_{im}u_m$, we can write this as

$$(\rho\omega^2\delta_{im} - \lambda_{iklm}k_k k_l)u_m = 0. \tag{23.2}$$

This is a set of three homogeneous equations of the first degree for the unknowns u_x, u_y, u_z. Such equations have non-zero solutions only if the determinant of the coefficients is zero. Thus we must have

$$|\lambda_{iklm}k_k k_l - \rho\omega^2\delta_{im}| = 0. \tag{23.3}$$

This is a cubic equation in ω^2. It has three roots, which are in general different. Each root gives the frequency as a function of the wave vector $\mathbf{k}$.†

Substituting each in turn in equation (23.2), we obtain equations giving the components of the corresponding displacement u_i (since the equations are homogeneous, of course, only the ratios of the three components u_i are obtained, and not their absolute values, so that all the u_i can be multiplied by an arbitrary constant).

The velocity of propagation of the wave (the *group velocity*) is given by the derivative of the frequency with respect to the wave vector. In an isotropic body, the frequency is proportional to the magnitude of $\mathbf{k}$, and so the direction of the velocity $\mathbf{U} = \partial\omega/\partial\mathbf{k}$ is the same as that of $\mathbf{k}$. In crystals this relation does not hold, and the direction of propagation of the wave is therefore not the same as that of its wave vector.

† In an isotropic body, equation (23.3) gives the result previously obtained: one root $\omega^2 = c_l^2k^2$ and two coincident roots $\omega^2 = c_t^2k^2$.

It is seen from equation (23.3) that ω is a homogeneous function, of degree one, of the components k_i; if the unknown quantity is taken as the ratio ω/k, the coefficients in the equation do not depend on k. Hence the velocity of propagation $\partial\omega/\partial\mathbf{k}$ is a homogeneous function, of degree zero, of k_i. Thus the velocity of a wave is a function of its direction, but not of its frequency.

Since there are three possible relations between ω and $\mathbf{k}$ for any direction in the crystal, there are in general three different velocities of propagation of elastic waves. These velocities are the same only in a few exceptional directions.

In an isotropic medium, purely longitudinal and purely transverse waves correspond to two different velocities of propagation. In a crystal, on the other hand, to each velocity of propagation there corresponds a wave in which the displacement vector has components both parallel and perpendicular to the direction of propagation.

Finally, we may notice the following. For any given wave vector $\mathbf{k}$ in a crystal there can be three waves, with different frequencies and velocities of propagation. It is easy to see that the displacement vectors $\mathbf{u}$ in these three waves are mutually perpendicular. For, when $\mathbf{k}$ is given, equation (23.3) may be regarded as determining the principal values $\rho\omega^2$ of a tensor of rank two, $\lambda_{iklm}k_k k_l$, which is symmetrical with respect to the suffixes i, m.† Equations (23.2) then give the principal axes of this tensor, which we know are mutually perpendicular.

PROBLEM

Determine the frequency as a function of the wave vector for elastic waves propagated in a crystal of the hexagonal system.

SOLUTION. The non-zero components of the tensor λ_{iklm} in the co-ordinates x, y, z are related to those in the co-ordinates ξ, η, z (see §10) by

$$\lambda_{xxxx} = \lambda_{yyyy} = a+b, \qquad \lambda_{xxyy} = a-b, \qquad \lambda_{xyxy} = b,$$

$$\lambda_{xxzz} = \lambda_{yyzz} = c, \qquad \lambda_{xzxz} = \lambda_{yzyz} = d, \qquad \lambda_{zzzz} = f,$$

where we have put

$$\lambda_{\xi\eta\xi\eta} = 4a, \qquad \lambda_{\xi\xi\eta\eta} = 8b, \qquad \lambda_{\xi\eta zz} = 2c, \qquad \lambda_{\xi z\eta z} = 2d.$$

The z-axis is along the sixth-order axis of symmetry; the directions of the x and y axes are arbitrary. We take the xz-plane such that it contains the wave vector $\mathbf{k}$. Then $k_x = k\sin\theta$, $k_y = 0$, $k_z = k\cos\theta$, where θ is the angle between $\mathbf{k}$ and the z-axis. Forming the equation (23.3) and solving it, we obtain three different dependences of ω on k:

$$\omega_1{}^2 = k^2(b\sin^2\theta + d\cos^2\theta)/\rho,$$

$$\omega_{2,3}{}^2 = \frac{k^2}{2\rho}\{(a+b)\sin^2\theta + f\cos^2\theta + d \pm \sqrt{([(a+b-d)\sin^2\theta + (d-f)\cos^2\theta]^2 + {}}$$

$$+4(c+d)^2\sin^2\theta\cos^2\theta)\}.$$

---

† By the symmetry of the tensor λ_{iklm}, we have $\lambda_{iklm}k_k k_l = \lambda_{kiml}k_k k_l = \lambda_{mlki}k_k k_l$. The latter expression differs from $\lambda_{mkli}k_k k_l$ only by the naming of the suffixes k and l, so that the tensor $\lambda_{iklm}k_k k_l$ has the symmetry stated.

§24. Surface waves

A particular kind of elastic waves are those propagated near the surface of a body without penetrating into it (*Rayleigh waves*). We write the equation of motion in the form (22.11) and (22.12):

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0, \tag{24.1}$$

where u is any component of the vectors $\mathbf{u}_l$, $\mathbf{u}_t$, and c is the corresponding velocity c_l or c_t, and seek solutions corresponding to these surface waves. The surface of the elastic medium is supposed plane and of infinite extent. We take this plane as the xy-plane; let the medium be in $z < 0$.

Let us consider a plane monochromatic surface wave propagated along the x-axis. Accordingly $u = e^{i(kx-\omega t)}f(z)$. Substituting this expression in (24.1), we obtain for the function $f(z)$ the equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) f.$$

If $k^2 - \omega^2/c^2 < 0$, this equation gives a periodic function f, i.e. we obtain an ordinary plane wave which is not damped inside the body. We must therefore suppose that $k^2 - \omega^2/c^2 > 0$. Then the solutions for f are

$$f(z) = \text{constant} \times \exp\left(\pm \sqrt{\left[k^2 - \frac{\omega^2}{c^2}\right]} z\right).$$

The solution with the minus sign would correspond to an unlimited increase in the deformation for $z \to -\infty$. This solution is clearly impossible, and so the plus sign must be taken.

Thus we have the following solution of the equations of motion:

$$u = \text{constant} \times e^{i(kx-\omega t)} e^{\kappa z}, \tag{24.2}$$

where

$$\kappa = \sqrt{(k^2 - \omega^2/c^2)}. \tag{24.3}$$

It corresponds to a wave which is exponentially damped towards the interior of the medium, i.e. is propagated only near the surface. The quantity κ determines the rapidity of the damping.

The true displacement vector $\mathbf{u}$ in the wave is the sum of the vectors $\mathbf{u}_l$ and $\mathbf{u}_t$, the components of each of which satisfy the equation (24.1) with $c = c_l$ for $\mathbf{u}_l$ and c_t for $\mathbf{u}_t$. For volume waves in an infinite medium, the two parts are independently propagated waves. For surface waves, however, this division into two independent parts is not possible, on account of the boundary conditions. The displacement vector $\mathbf{u}$ must be a definite linear combination of the vectors $\mathbf{u}_l$ and $\mathbf{u}_t$. It should also be mentioned that these latter vectors have no longer the simple significance of the displacement components parallel and perpendicular to the direction of propagation.

To determine the linear combination of the vectors $\mathbf{u}_l$ and $\mathbf{u}_t$ which gives the true displacement $\mathbf{u}$, we must use the conditions at the boundary of the body. These give a relation between the wave vector $\mathbf{k}$ and the frequency ω, and therefore the velocity of propagation of the wave. At the free surface we must have $\sigma_{ik}n_k = 0$. Since the normal vector $\mathbf{n}$ is parallel to the z-axis, it follows that $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, whence

$$u_{xz} = 0, \qquad u_{yz} = 0, \qquad \sigma(u_{xx}+u_{yy})+(1-\sigma)u_{zz} = 0. \qquad (24.4)$$

Since all quantities are independent of the co-ordinate y, the second of these conditions gives

$$u_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) = \tfrac{1}{2}\partial u_y/\partial z = 0.$$

Using (24.2), we therefore have

$$u_y = 0. \qquad (24.5)$$

Thus the displacement vector $\mathbf{u}$ in a surface wave is in a plane through the direction of propagation perpendicular to the surface.

The transverse part $\mathbf{u}_t$ of the wave must satisfy the condition (22.8) div $\mathbf{u}_t = 0$, or

$$\frac{\partial u_{tx}}{\partial x}+\frac{\partial u_{tz}}{\partial z} = 0.$$

The dependence of u_{tx} and u_{tz} on x and z is determined by the factor $e^{ikx+\kappa_t z}$, where κ_t is given by the expression (24.3) with $c = c_t$, i.e.

$$\kappa = \sqrt{(k^2-\omega^2/c_t^2)}.$$

Hence the above condition leads to the equation

$$iku_{tx}+\kappa_t u_{tz} = 0, \text{ or } u_{tx}/u_{tz} = -\kappa_t/ik.$$

Thus we can write

$$u_{tx} = \kappa_t a e^{ikx+\kappa_t z-i\omega t}, \qquad u_{tz} = -ika e^{ikx+\kappa_t z-i\omega t}, \qquad (24.6)$$

where a is some constant.

The longitudinal part $\mathbf{u}_l$ satisfies the condition (22.9) **curl** $\mathbf{u}_l = 0$, or

$$\frac{\partial u_{lx}}{\partial z}-\frac{\partial u_{lz}}{\partial x} = 0,$$

whence

$$iku_{lz}-\kappa_l u_{lx} = 0 \quad (\kappa_l = \sqrt{[k^2-\omega^2/c_l^2]}).$$

Thus we must have

$$u_{lx} = kbe^{ikx+\kappa_l z-i\omega t}, \qquad u_{lz} = -i\kappa_l be^{ikx+\kappa_l z-i\omega t}, \qquad (24.7)$$

where b is a constant.

We now use the first and third conditions (24.4). Expressing u_{ik} in terms of the derivatives of u_i, and using the velocities c_l, c_t, we can write these conditions as

$$\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x} = 0,$$

$$c_l^2\frac{\partial u_z}{\partial z}+(c_l^2-2c_t^2)\frac{\partial u_x}{\partial x} = 0. \tag{24.8}$$

Here we must substitute $u_x = u_{lx}+u_{tx}$, $u_z = u_{lz}+u_{tz}$. The result is that the first condition (24.8) gives

$$a(k^2+\kappa_t^2)+2bk\kappa_l = 0. \tag{24.9}$$

The second condition leads to the equation

$$2ac_t^2\kappa_t k+b[c_l^2(\kappa_l^2-k^2)+2c_t^2k^2] = 0.$$

Dividing this equation by c_t^2 and substituting

$$\kappa_l^2-k^2 = -\omega^2/c_l^2 = -(k^2-\kappa_t^2)c_t^2/c_l^2,$$

we can write it as

$$2a\kappa_t k+b(k^2+\kappa_t^2) = 0. \tag{24.10}$$

The condition for the two homogeneous equations (24.9) and (24.10) to be compatible is $(k^2+\kappa_t^2)^2 = 4k^2\kappa_t\kappa_l$ or, squaring and substituting the values of κ_t^2 and κ_l^2,

$$\left(2k^2-\frac{\omega^2}{c_t^2}\right)^4 = 16k^4\left(k^2-\frac{\omega^2}{c_t^2}\right)\left(k^2-\frac{\omega^2}{c_l^2}\right). \tag{24.11}$$

From this equation we obtain the relation between ω and k. It is convenient to put

$$\omega = c_t k\xi; \tag{24.12}$$

k^8 then cancels from both sides of the equation, and, expanding, we obtain for ξ the equation

$$\xi^6-8\xi^4+8\xi^2\left(3-2\frac{c_t^2}{c_l^2}\right)-16\left(1-\frac{c_t^2}{c_l^2}\right) = 0. \tag{24.13}$$

Hence we see that ξ depends only on the ratio c_t/c_l, which is a constant characteristic of any given substance and in turn depends only on POISSON's ratio:

$$c_t/c_l = \sqrt{\{(1-2\sigma)/2(1-\sigma)\}}.$$

The quantity ξ must, of course, be real and positive, and $\xi < 1$ (so that κ_t and κ_l are real). Equation (24.13) has only one root satisfying these conditions, and so a single value of ξ is obtained for any given value of c_t/c_l.

Thus, for both surface waves and volume waves, the frequency is proportional to the wave number. The proportionality coefficient is the velocity of propagation of the wave,

$$U = c_t\xi. \tag{24.14}$$

This gives the velocity of propagation of surface waves in terms of the velocities c_t and c_l of the transverse and longitudinal volume waves. The ratio of the amplitudes of the transverse and longitudinal parts of the wave is given in terms of ξ by the formula

$$\frac{a}{b} = -\frac{2-\xi^2}{2\sqrt{(1-\xi^2)}}. \tag{24.15}$$

The ratio c_t/c_l actually varies from $1/\sqrt{2}$ to 0 for various substances, corresponding to the variation of σ from 0 to $\frac{1}{2}$; ξ then varies from 0·874 to 0·955. Fig. 21 shows a graph of ξ as a function of σ.

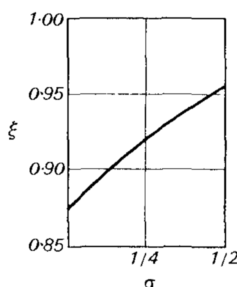


Fig. 21

PROBLEM

A plane-parallel slab of thickness h (medium 1) lies on an elastic half-space (medium 2). Determine the frequency as a function of the wave number for transverse waves in the slab whose direction of oscillation is parallel to its boundaries.

Solution. We take the plane separating the slab from the half-space as the xy-plane, the half-space being in $z < 0$ and the slab in $0 \leqslant z \leqslant h$. In the slab we have

$$u_{x1} = u_{z1} = 0, \qquad u_{y1} = f(z)e^{i(kx-\omega t)},$$

and in medium 2 a damped wave:

$$u_{x2} = u_{z2} = 0, \qquad u_{y2} = Ae^{\kappa_2 z}e^{i(kx-\omega t)}, \qquad \kappa_2 = \sqrt{(k^2-\omega^2/c_{t2}^2)}.$$

For the function $f(z)$ we have the equation

$$f'' + \kappa_1^2 f = 0, \qquad \kappa_1 = \sqrt{(\omega^2/c_{t1}^2-k^2)}$$

(we shall see below that $\kappa_1^2 > 0$), whence $f(z) = B\sin\kappa_1 z + C\cos\kappa_1 z$. At the free surface of the slab ($z = h$) we must have $\sigma_{zy} = 0$, i.e. $\partial u_{y1}/\partial z = 0$. At the boundary between the two media ($z = 0$) the conditions are $u_{y1} = u_{y2}$, $\mu_1 \partial u_{y1}/\partial z = \mu_2 \partial u_{y2}/\partial z$, μ_1 and μ_2 being the moduli of rigidity for the two media. From these conditions we find three equations for A, B, C, and the compatibility condition is $\tan\kappa_1 h = \mu_2\kappa_2/\mu_1\kappa_1$. This equation gives ω as an implicit function of k; it has solutions only for real κ_1 and κ_2, and so $c_{t2} > \omega/k > c_{t1}$. Hence we see that such waves can be propagated only if $c_{t2} > c_{t1}$.

§25. Vibration of rods and plates

Waves propagated in thin rods and plates are fundamentally different from those propagated in a medium infinite in all directions. Here we are speaking of waves of length large compared with the thickness of the rod or plate. If the wavelength is small compared with this thickness, the rod or plate is effectively infinite in all directions as regards the propagation of the wave, and we return to the results obtained for infinite media.

Waves in which the oscillations are parallel to the axis of the rod or the plane of the plate must be distinguished from those in which they are perpendicular to it. We shall begin by studying longitudinal waves in rods.

A longitudinal deformation of the rod (uniform over any cross-section), with no external force on the sides of the rod, is a simple extension or compression. Thus longitudinal waves in a rod are simple extensions or compressions propagated along its length. In a simple extension, however, only the component σ_{zz} of the stress tensor (the z-axis being along the rod) is different from zero; it is related to the strain tensor by $\sigma_{zz} = Eu_{zz} = E\partial u_z/\partial z$ (see §5). Substituting this in the general equation of motion $\rho\ddot{u}_z = \partial\sigma_{zk}/\partial x_k$, we find

$$\frac{\partial^2 u_z}{\partial z^2} - \frac{\rho}{E}\frac{\partial^2 u_z}{\partial t^2} = 0. \tag{25.1}$$

This is the equation of longitudinal vibrations in rods. We see that it is an ordinary wave equation. The velocity of propagation of longitudinal waves in rods is

$$\sqrt{(E/\rho)}. \tag{25.2}$$

Comparing this with the expression (22.4) for c_l, we see that it is less than the velocity of propagation of longitudinal waves in an infinite medium.

Let us now consider longitudinal waves in thin plates. The equations of motion for such vibrations can be written down at once by substituting $-\rho h\partial^2 u_x/\partial t^2$ and $-\rho h\partial^2 u_y/\partial t^2$ for P_x and P_y in the equilibrium equations (13.4):

$$\begin{aligned}\frac{\rho}{E}\frac{\partial^2 u_x}{\partial t^2} &= \frac{1}{1-\sigma^2}\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 u_x}{\partial y^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 u_y}{\partial x\partial y},\\ \frac{\rho}{E}\frac{\partial^2 u_y}{\partial t^2} &= \frac{1}{1-\sigma^2}\frac{\partial^2 u_y}{\partial y^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 u_x}{\partial x\partial y}.\end{aligned} \tag{25.3}$$

We take the case of a plane wave propagated along the x-axis, i.e. a wave in which the deformation depends only on the co-ordinate x, and not on y. Then equations (25.3) are much simplified, becoming

$$\frac{\partial^2 u_x}{\partial t^2} - \frac{E}{\rho(1-\sigma^2)}\frac{\partial^2 u_x}{\partial x^2} = 0, \quad \frac{\partial^2 u_y}{\partial t^2} - \frac{E}{2\rho(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2} = 0. \tag{25.4}$$

We thus again obtain wave equations. The coefficients are different for u_x and u_y. The velocity of propagation of a wave with oscillations parallel to the direction of propagation (u_x) is

$$\sqrt{[E/\rho(1-\sigma^2)]}. \tag{25.5}$$

The velocity for a wave (u_y) with oscillations perpendicular to the direction of propagation (but still in the plane of the plate) is equal to the velocity c_t of transverse waves in an infinite medium.

Thus we see that longitudinal waves in rods and plates are of the same nature as in an infinite medium, only the velocity being different; as before, it is independent of the frequency. Entirely different results are obtained for *bending waves* in rods and plates, for which the oscillations are in a direction perpendicular to the axis of the rod or the plane of the plate, i.e. involve bending.

The equations for free oscillations of a plate can be written down at once from the equilibrium equation (12.5). To do so, we must replace $-P$ by the acceleration $\ddot{\zeta}$ multiplied by the mass ρh per unit area of the plate. This gives

$$\rho\frac{\partial^2 \zeta}{\partial t^2} + \frac{Eh^2}{12(1-\sigma^2)}\triangle^2\zeta = 0, \tag{25.6}$$

where $\triangle$ is the two-dimensional Laplacian.

Let us consider a monochromatic elastic wave, and accordingly seek a solution of equation (25.6) in the form

$$\zeta = \text{constant} \times e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{25.7}$$

where the wave vector $\mathbf{k}$ has, of course, only two components, k_x and k_y. Substituting in (25.6), we obtain the equation

$$-\rho\omega^2 + Eh^2k^4/12(1-\sigma^2) = 0.$$

Hence we have the following relation between the frequency and the wave number:

$$\omega = k^2\sqrt{\{Eh^2/12\rho(1-\sigma^2)\}}. \tag{25.8}$$

Thus the frequency is proportional to the square of the wave number, whereas in waves in an infinite medium it is proportional to the wave number itself.

Knowing the relation between the frequency and the wave number, we can determine the velocity of propagation of the wave from the formula

$$\mathbf{U} = \partial\omega/\partial\mathbf{k}.$$

The derivatives of k^2 with respect to the components k_x, k_y are respectively $2k_x$, $2k_y$. The velocity of propagation of the wave is therefore

$$\mathbf{U} = \mathbf{k}\sqrt{\{Eh^2/3\rho(1-\sigma^2)\}}. \qquad (25.9)$$

It is proportional to the wave vector, and not a constant as it is for waves in a medium infinite in three dimensions.†

Similar results are obtained for bending waves in thin rods. The bending deflections of the rod are supposed small. The equations of motion are obtained by replacing $-K_x$ and $-K_y$ in the equations of equilibrium for a slightly bent rod (20.4) by the product of the acceleration $\ddot{X}$ or $\ddot{Y}$ and the mass ρS per unit length of the rod (S being its cross-sectional area). Thus

$$\rho S\ddot{X} = EI_y\partial^4 X/\partial z^4, \qquad \rho S\ddot{Y} = EI_x\partial^4 Y/\partial z^4. \qquad (25.10)$$

We again seek solutions of these equations in the form

$$X = \text{constant}\times e^{i(kz-\omega t)}, \qquad Y = \text{constant}\times e^{i(kz-\omega t)}.$$

Substituting in (25.10), we obtain the following relations between the frequency and the wave number:

$$\omega = k^2\sqrt{(EI_y/\rho S)}, \qquad \omega = k^2\sqrt{(EI_x/\rho S)}, \qquad (25.11)$$

for vibrations in the x and y directions respectively. The corresponding velocities of propagation are

$$U^{(x)} = 2k\sqrt{(EI_y/\rho S)}, \qquad U^{(y)} = 2k\sqrt{(EI_x/\rho S)}. \qquad (25.12)$$

Finally, there is a particular case of vibration of rods called ***torsional vibration***. The corresponding equations of motion are derived by equating $C\partial\tau/\partial z$ (see §18) to the time derivative of the angular momentum of the rod per unit length. This angular momentum is $\rho I\partial\phi/\partial t$, where $\partial\phi/\partial t$ is the angular velocity (ϕ being the angle of rotation of the cross-section considered) and $I = \int(x^2+y^2)\,df$ is the moment of inertia of the cross-section about its centre of mass; for pure torsional vibration each cross-section of the rod performs rotary vibrations about its centre of mass, which remains at rest. Putting $\tau = \partial\phi/\partial z$, we obtain the equation of motion in the form

$$C\partial^2\phi/\partial z^2 = \rho I\partial^2\phi/\partial t^2. \qquad (25.13)$$

† The wave number $k = 2\pi/\lambda$, where λ is the wavelength. Hence the velocity of propagation should increase without limit as λ tends to zero. This physically impossible result is obtained because formula (25.9) is not valid for short waves.

Hence we see that the velocity of propagation of torsional oscillations along the rod is

$$\sqrt{(C/\rho I)}. \tag{25.14}$$

PROBLEMS

PROBLEM 1. Determine the characteristic frequencies of longitudinal vibrations of a rod of length l, with one end fixed and the other free.

SOLUTION. At the fixed end ($z = 0$) we must have $u_z = 0$, and at the free end ($z = l$) $\sigma_{zz} = Eu_{zz} = 0$, i.e. $\partial u_z/\partial z = 0$. We seek a solution of equation (25.1) in the form

$$u_z = A\cos(\omega t+\alpha)\sin kz,$$

where $k = \omega\sqrt{(\rho/E)}$. From the condition at $z = l$ we have $\cos kl = 0$, whence the characteristic frequencies are

$$\omega = \sqrt{(E/\rho)}(2n+1)\pi/2l,$$

n being any integer.

PROBLEM 2. The same as Problem 1, but for a rod with both ends free or both fixed.

SOLUTION. In either case $\omega = \sqrt{(E/\rho)}\, n\pi/l$.

PROBLEM 3. Determine the characteristic frequencies of vibration of a string of length l.

SOLUTION. The equation of motion of the string is

$$\frac{\partial^2 X}{\partial z^2} - \frac{\rho S}{T}\frac{\partial^2 X}{\partial t^2} = 0;$$

cf. the equilibrium equation (20.17). The boundary conditions are that $X = 0$ for $z = 0$ and l. The characteristic frequencies are $\omega = \sqrt{(\rho S/T)}n\pi/l$.

PROBLEM 4. Determine the characteristic transverse vibrations of a rod (of length l) with clamped ends.

SOLUTION. Equation (25.10), on substituting $X = X_0(z)\cos(\omega t+\alpha)$, becomes

$$\mathrm{d}^4 X_0/\mathrm{d}z^4 = \kappa^4 X_0,$$

where $\kappa^4 = \omega^2\rho S/EI_y$. The general integral of this equation is

$$X_0 = A\cos\kappa z+B\sin\kappa z+C\cosh\kappa z+D\sinh\kappa z.$$

The constants A, B, C and D are determined from the boundary conditions that $X = \mathrm{d}X/\mathrm{d}z = 0$ for $z = 0$ and l. The result is

$$X_0 = A\{(\sin\kappa l-\sinh\kappa l)(\cos\kappa z-\cosh\kappa z)- \\ -(\cos\kappa l-\cosh\kappa l)(\sin\kappa z-\sinh\kappa z)\},$$

and the equation $\cos\kappa l\,\cosh\kappa l = 1$, the roots of which give the characteristic frequencies The smallest characteristic frequency is

$$\omega_{\min} = \frac{22{\cdot}4}{l^2}\sqrt{\frac{EI_y}{\rho S}}.$$

PROBLEM 5. The same as Problem 4, but for a rod with supported ends.

SOLUTION. In the same way as in Problem 4, we obtain $X_0 = A \sin \kappa z$, and the frequencies are given by $\sin \kappa l = 0$, i.e. $\kappa = n\pi/l$ $(n = 1, 2, \ldots)$. The smallest frequency is

$$\omega_{\min} = \frac{9{\cdot}87}{l^2}\sqrt{\frac{EI_y}{\rho S}}.$$

PROBLEM 6. The same as Problem 4, but for a rod with one end clamped and the other free.

SOLUTION. We have for the displacement

$$X_0 = A\{(\cos \kappa l + \cosh \kappa l)(\cos \kappa z - \cosh \kappa z) + (\sin \kappa l - \sinh \kappa l)(\sin \kappa z - \sinh \kappa z)\}$$

(the clamped end being at $z = 0$ and the free end at $z = l$), and for the characteristic frequencies the equation $\cos \kappa l \cosh \kappa l + 1 = 0$. The smallest frequency is

$$\omega_{\min} = \frac{3{\cdot}52}{l^2}\sqrt{\frac{EI_y}{\rho S}}.$$

PROBLEM 7. Determine the characteristic vibrations of a rectangular plate of sides a and b, with its edges supported.

SOLUTION. Equation (25.6), on substituting $\zeta = \zeta_0(x, y)\cos(\omega t + \alpha)$, becomes

$$\triangle^2 \zeta_0 - \kappa^4 \zeta_0 = 0,$$

where $\kappa^4 = 12\rho(1-\sigma^2)\omega^2/Eh^2$. We take the co-ordinate axes along the sides of the plate. The boundary conditions (12.11) become $\zeta = \partial^2\zeta/\partial x^2 = 0$ for $x = 0$ and a,

$$\zeta = \partial^2\zeta/\partial y^2 = 0$$

for $y = 0$ and b. The solution which satisfies these conditions is

$$\zeta_0 = A \sin(m\pi x/a)\sin(n\pi y/b),$$

where m and n are integers. The frequencies are given by

$$\omega = h\sqrt{\frac{E}{12\rho(1-\sigma^2)}}\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right].$$

PROBLEM 8. Determine the characteristic frequencies for the vibration of a rectangular membrane of sides a and b.

SOLUTION. The equation for the vibration of a membrane is $T\triangle\zeta = \rho h\ddot{\zeta}$; cf. the equilibrium equation (14.9). The edges of the membrane must be fixed, so that $\zeta = 0$. The corresponding solution for a rectangular membrane is

$$\zeta = A \sin(m\pi x/a)\sin(n\pi y/b)\cos \omega t,$$

where the characteristic frequencies are given by

$$\omega^2 = \frac{T\pi^2}{\rho h}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right),$$

m and n being integers.

PROBLEM 9. Determine the velocity of propagation of torsional vibrations in a rod whose cross-section is a circle, an ellipse, or an equilateral triangle, and in a rod in the form of a long thin rectangular plate.

SOLUTION. For a circular cross-section of radius R, the moment of inertia is $I = \frac{1}{2}\pi R^4$; C is given in §16, Problem 1, and we find the velocity to be $\sqrt{(\mu/\rho)}$, which is the same as the velocity c_t.

Similarly (using the results of §16, Problems 2 to 4), we find for a rod of elliptical cross-section the velocity $[2ab/(a^2+b^2)]\sqrt{(\mu/\rho)}$, for one with an equilateral triangular cross-section $\sqrt{(3\mu/5\rho)}$, and for one which is a long rectangular plate $(2h/d)\sqrt{(\mu/\rho)}$. All these are less than c_t.

PROBLEM 10. The surface of an incompressible fluid of infinite depth is covered by a thin elastic plate. Determine the relation between the wave number and the frequency for waves which are simultaneously propagated in the plate and near the surface of the fluid.

SOLUTION. We take the plane of the plate as $z = 0$, and the x-axis in the direction of propagation of the wave; let the fluid be in $z < 0$. The equation of motion of the plate alone would be

$$\rho_0 h\frac{\partial^2\zeta}{\partial t^2} = -\frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^4\zeta}{\partial x^4},$$

where ρ_0 is the volume density of the plate. When the fluid is present, the right-hand side of this equation must also include the force exerted by the fluid on unit area of the plate, i.e. the pressure p of the fluid. The pressure in the wave, however, can be expressed in terms of the velocity potential by $p = -\rho\partial\phi/\partial t$ (we neglect gravity). Hence we obtain

$$\rho_0 h\frac{\partial^2\zeta}{\partial t^2} = -\frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^4\zeta}{\partial x^4} - \left[\rho\frac{\partial\phi}{\partial t}\right]_{z=0}. \tag{1}$$

Next, the normal component of the fluid velocity at the surface must be equal to that of the plate, whence

$$\partial\zeta/\partial t = [\partial\phi/\partial z]_{z=0}. \tag{2}$$

The potential ϕ must satisfy everywhere in the fluid the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2} = 0. \tag{3}$$

We seek ζ in the form of a travelling wave $\zeta = \zeta_0 e^{ikx-i\omega t}$; accordingly, we take as the solution of equation (3) the surface wave $\phi = \phi_0 e^{i(kx-\omega t)}e^{kz}$, which is damped in the interior of the fluid. Substituting these expressions in (1) and (2), we obtain two equations for ϕ_0 and ζ_0, and the compatibility condition is

$$\omega^2 = \frac{Eh^3}{12(1-\sigma^2)}\cdot\frac{k^5}{\rho+h\rho_0 k}.$$

§26. Anharmonic vibrations

The whole of the theory of elastic vibrations given above is approximate to the extent that any theory of elasticity is so which is based on HOOKE's law. It should be recalled that the theory begins from an expansion of the elastic energy as a power series with respect to the strain tensor, which includes terms up to and including the second order. The components of

the stress tensor are then linear functions of those of the strain tensor, and the equations of motion are linear.

The most characteristic property of elastic waves in this approximation is that any wave can be obtained by simple superposition (i.e. as a linear combination) of separate monochromatic waves. Each of these is propagated independently, and could exist by itself without involving any other motion. We may say that the various monochromatic waves which are simultaneously propagated in a single medium do not interact with one another.

These properties, however, no longer hold in subsequent approximations. The effects which appear in these approximations, though small, may be of importance as regards certain phenomena. They are usually called *anharmonic effects*, since the corresponding equations of motion are non-linear and do not admit simple periodic (harmonic) solutions.

We shall consider here anharmonic effects of the third order, arising from terms in the elastic energy which are cubic in the strains. It would be too cumbersome to write out the corresponding equations of motion in their general form. However, the nature of the resulting effects can be ascertained as follows. The cubic terms in the elastic energy give quadratic terms in the stress tensor, and therefore in the equations of motion. Let us suppose that all the linear terms in these equations are on the left-hand side, and all the quadratic terms on the right-hand side. Solving these equations by the method of successive approximations, we omit the quadratic terms in the first approximation. This leaves the ordinary linear equations, whose solution $\mathbf{u}_0$ can be put in the form of a superposition of monochromatic travelling waves: constant $\times e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, with definite relations between ω and $\mathbf{k}$. On going to the second approximation, we must put $\mathbf{u} = \mathbf{u}_0+\mathbf{u}_1$ and retain only the terms in $\mathbf{u}_0$ on the right-hand sides of the equations (the quadratic terms). Since $\mathbf{u}_0$, by definition, satisfies the homogeneous linear equations obtained by putting the right-hand sides equal to zero, the terms in $\mathbf{u}_0$ on the left hand sides will cancel. The result is a set of inhomogeneous linear equations for the components of the vector $\mathbf{u}_1$, where the right-hand sides contain only known functions of the co-ordinates and time. These functions, which are obtained by substituting $\mathbf{u}_0$ for $\mathbf{u}$ in the right-hand sides of the original equations, are sums of terms each of which is proportional to

$$e^{i[(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}-(\omega_1-\omega_2)t]}$$

or

$$e^{i[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}-(\omega_1+\omega_2)t]},$$

where ω_1, ω_2, $\mathbf{k}_1$, $\mathbf{k}_2$ are the frequencies and wave vectors of any two monochromatic waves in the first approximation.

A particular integral of linear equations of this type is a sum of terms containing similar exponential factors to those in the free terms (the right-hand sides) of the equations, with suitably chosen coefficients. Each such term corresponds to a travelling wave with frequency $\omega_1 \pm \omega_2$ and wave

vector $\mathbf{k}_1 \pm \mathbf{k}_2$. Frequencies equal to the sum or difference of the frequencies of the original waves are called *combination frequencies*.

Thus the anharmonic effects in the third order have the result that the set of fundamental monochromatic waves (with frequencies ω_1, ω_2, ... and wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, ...) has superposed on it other "waves" of small intensity, whose frequencies are the combination frequencies such as $\omega_1 \pm \omega_2$, and whose wave vectors are such as $\mathbf{k}_1 \pm \mathbf{k}_2$. We call these "waves" in quotation marks because they are a correction effect and cannot exist alone except in certain special cases (see below). The values $\omega_1 \pm \omega_2$ and $\mathbf{k}_1 \pm \mathbf{k}_2$ do not in general satisfy the relations which hold between the frequencies and wave vectors for ordinary monochromatic waves.

It is clear, however, that there may happen to be particular values of ω_1, $\mathbf{k}_1$ and ω_2, $\mathbf{k}_2$ such that one of the relations for monochromatic waves in the medium considered also holds for $\omega_1 + \omega_2$ and $\mathbf{k}_1 + \mathbf{k}_2$ (for definiteness, we shall discuss sums and not differences). Putting $\omega_3 = \omega_1 + \omega_2$, $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$, we can say that, mathematically, ω_3 and $\mathbf{k}_3$ then correspond to waves which satisfy the homogeneous linear equations of motion (with zero on the right-hand side) in the first approximation. If the right-hand sides in the second approximation contain terms proportional to $e^{i(\mathbf{k}_3 \cdot \mathbf{r} - \omega_3 t)}$, then a particular integral will be a wave with the same frequency and an amplitude which increases indefinitely with time.

Thus the superposition of two monochromatic waves with values of ω_1, $\mathbf{k}_1$ and ω_2, $\mathbf{k}_2$ whose sum ω_3, $\mathbf{k}_3$ satisfies the above condition leads, by the anharmonic effects, to resonance: a new monochromatic wave (with parameters ω_3, $\mathbf{k}_3$) is formed, whose amplitude increases with time and eventually is no longer small. It is evident that, if a wave with ω_3, $\mathbf{k}_3$ is formed on superposition of those with ω_1, $\mathbf{k}_1$ and ω_2, $\mathbf{k}_2$, then the superposition of waves with ω_1, $\mathbf{k}_1$ and ω_3, $\mathbf{k}_3$ will also give a resonance with $\omega_2 = \omega_3 - \omega_1$, $\mathbf{k}_2 = \mathbf{k}_3 - \mathbf{k}_1$, and similarly ω_2, $\mathbf{k}_2$ and ω_3, $\mathbf{k}_3$ lead to ω_1, $\mathbf{k}_1$.

In particular, for an isotropic body ω and $\mathbf{k}$ are related by $\omega = c_t k$ or $\omega = c_l k$, with $c_l > c_t$. It is easy to see in which cases either of these relations can hold for each of the three combinations

$$\omega_1, \mathbf{k}_1; \quad \omega_2, \mathbf{k}_2; \quad \omega_3 = \omega_1 + \omega_2, \quad \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2.$$

If $\mathbf{k}_1$ and $\mathbf{k}_2$ are not in the same direction, $k_3 < k_1 + k_2$, and so it is clear that resonance can then occur only in the following two cases: (1) the waves with ω_1, $\mathbf{k}_1$ and ω_2, $\mathbf{k}_2$ are transverse and that with ω_3, $\mathbf{k}_3$ longitudinal; (2) one of the waves with ω_1, $\mathbf{k}_1$ and ω_2, $\mathbf{k}_2$ is transverse and the other longitudinal, and that with ω_3, $\mathbf{k}_3$ is longitudinal. If the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are in the same direction, however, resonance is possible when all three waves are longitudinal or all three are transverse.

The anharmonic effect involving resonance occurs not only when several monochromatic waves are superposed, but also when there is only one wave,

with parameters ω_1, $\mathbf{k}_1$. In this case the right-hand sides of the equations of motion contain terms proportional to $e^{2i(\mathbf{k}_1\cdot\mathbf{r}-\omega_1 t)}$. If ω_1 and $\mathbf{k}_1$ satisfy the usual condition, however, then $2\omega_1$ and $2\mathbf{k}_1$ do so too, since this condition is homogeneous and of degree one. Thus the anharmonic effect results in the appearance, besides the monochromatic waves with ω_1, $\mathbf{k}_1$ previously obtained, of waves with $2\omega_1$, $2\mathbf{k}_1$, i.e. with twice the frequency and twice the wave vector, and amplitude increasing with time.

Finally, we may briefly discuss how we can set up the equations of motion, allowing for the anharmonic terms. The strain tensor must now be given by the complete expression (1.3):

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}+\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right), \tag{26.1}$$

in which the terms quadratic in u_i can not be neglected. Next, the general expression for the energy density† $\mathscr{E}$, in bodies having a given symmetry, must be written as a scalar formed from the components of the tensor u_{ik} and some constant tensors characteristic of the substance involved; this scalar will contain terms up to a given power of u_{ik}. Substituting the expression (26.1) for u_{ik} and omitting terms in u_i of higher orders than that power, we find the energy $\mathscr{E}$ as a function of the derivatives $\partial u_i/\partial x_k$ to the required accuracy.

In order to obtain the equations of motion, we notice the following result. The variation $\delta\mathscr{E}$ may be written

$$\delta\mathscr{E} = \frac{\partial\mathscr{E}}{\partial(\partial u_i/\partial x_k)}\delta\frac{\partial u_i}{\partial x_k},$$

or, putting

$$\sigma_{ik} = \frac{\partial\mathscr{E}}{\partial(\partial u_i/\partial x_k)}, \tag{26.2}$$

$$\delta\mathscr{E} = \sigma_{ik}\frac{\partial\delta u_i}{\partial x_k} = \frac{\partial}{\partial x_k}(\sigma_{ik}\delta u_i)-\delta u_i\frac{\partial\sigma_{ik}}{\partial x_k}.$$

The coefficients of $-\delta u_i$ are the components of the force per unit volume of the body. They formally appear the same as before, and so the equations of motion can again be written

$$\rho_0\ddot{u}_i = \partial\sigma_{ik}/\partial x_k, \tag{26.3}$$

† We here use the internal energy $\mathscr{E}$, and not the free energy F, since adiabatic vibrations are involved.

where ρ_0 is the density of the undeformed body, and the components of the tensor σ_{ik} are now given by (26.2), with $\mathscr{E}$ correct to the required accuracy. The tensor σ_{ik} is no longer symmetrical.†

PROBLEM

Write down the general expression for the elastic energy of an isotropic body in the third approximation.

SOLUTION. From the components of a symmetrical tensor of rank two we can form two quadratic scalars (u_{ik}^2 and u_{ll}^2) and three cubic scalars (u_{ll}^3, $u_{ll}u_{ik}^2$ and $u_{ik}u_{il}u_{kl}$). Hence the most general scalar containing terms quadratic and cubic in u_{ik}, with scalar coefficients (since the body is isotropic), is

$$\mathscr{E} = \mu u_{ik}^2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)u_{ll}^2 + \tfrac{1}{3}Au_{ik}u_{il}u_{kl} + Bu_{ik}^2u_{ll} + \tfrac{1}{3}Cu_{ll}^3;$$

the coefficients of u_{ik}^2 and u_{ll}^2 have been expressed in terms of the moduli of compression and rigidity, and A, B, C are three new constants. Substituting the expression (26.1) for u_{ik} and retaining terms up to and including the third order, we find the elastic energy to be

$$\mathscr{E} = \tfrac{1}{4}\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)\left(\frac{\partial u_l}{\partial x_l}\right)^2 +$$
$$+ (\mu + \tfrac{1}{4}A)\frac{\partial u_i}{\partial x_k}\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k} + (\tfrac{1}{2}B + \tfrac{1}{2}K - \tfrac{1}{3}\mu)\frac{\partial u_l}{\partial x_l}\left(\frac{\partial u_i}{\partial x_k}\right)^2 +$$
$$+ \tfrac{1}{12}A\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_l}\frac{\partial u_l}{\partial x_i} + \tfrac{1}{2}B\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_l} + \tfrac{1}{3}C\left(\frac{\partial u_l}{\partial x_l}\right)^3.$$

† It should be emphasised that σ_{ik} is no longer the momentum flux density (the stress tensor). In the ordinary theory this interpretation was derived by integrating the body force density $\partial\sigma_{ik}/\partial x_k$ over the volume of the body. This derivation depended on the fact that, in performing the integration, we made no distinction between the co-ordinates of points in the body before and after the deformation. In subsequent approximations, however, this distinction must be made, and the surface bounding the region of integration is not the same as the actual surface of the region considered after the deformation.

It has been shown in §2 that the symmetry of the tensor σ_{ik} is due to the conservation of angular momentum. This result no longer holds, since the angular momentum density is not $x_i\dot{u}_k - x_k\dot{u}_i$ but $(x_i + u_i)\dot{u}_k - (x_k + u_k)\dot{u}_i$.

CHAPTER IV

DISLOCATIONS†

§27. Elastic deformations in the presence of a dislocation

Elastic deformations in a crystal may arise not only by the action of external forces on it but also because of internal structural defects present in the crystal. The principal type of defect that influences the mechanical properties of crystals is called a *dislocation*. The study of the properties of dislocations on the atomic or microscopic scale is not, of course, within the scope of this book; we shall here consider only purely macroscopic aspects of the phenomenon as it affects elasticity theory. For a better understanding of the physical significance of the relations obtained, however, we shall first give two simple examples to show what is the nature of dislocation defects as regards the structure of the crystal lattice.

Let us imagine that an "extra" half-plane is put into a crystal lattice of which a cross-section is shown in Fig. 22; in this diagram, the added half-plane

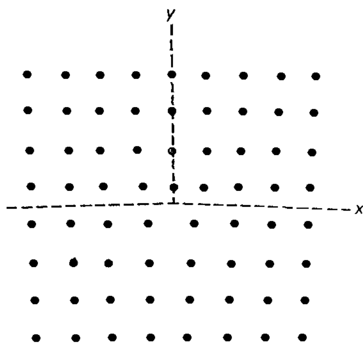


Fig. 22

is the upper half of the yz-plane. The edge of this half-plane (the z-axis, at right angles to the plane of the diagram) is then called an *edge dislocation*. In the immediate neighbourhood of the dislocation the crystal lattice is greatly distorted, but even at a distance of a few lattice periods the crystal planes fit together in an almost regular manner. The deformation nevertheless exists even far from the dislocation. It is clearly seen on going round a closed circuit of lattice points in the xy-plane, with the origin within the circuit: if the

† This chapter was written jointly with A. M. Kosevich.

displacement of each point from its position in the ideal lattice is denoted by the vector **u**, the total increment of this vector around the circuit will not be zero, but equals one lattice period in the x-direction.

Another type of dislocation may be visualised as the result of "cutting" the lattice along a half-plane and then shifting the parts of the lattice on either side of the cut in opposite directions to a distance of one lattice period parallel to the edge of the cut (then called a *screw dislocation*). Such a dislocation converts the lattice planes into a helicoidal surface, like a spiral staircase without the steps. In a complete circuit round the dislocation line (the axis of the helicoidal surface) the lattice point displacement vector increment is one lattice period along that axis. Figure 23 shows a diagram of such a cut.

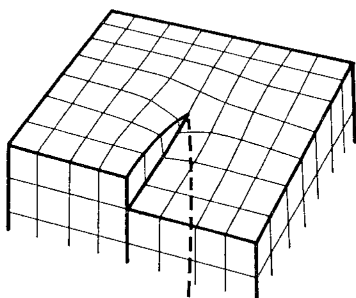


Fig. 23

Macroscopically, a dislocation deformation of a crystal regarded as a continuous medium has the following general property; after a passage round any closed contour L which encloses the dislocation line D, the elastic displacement vector **u** receives a certain finite increment **b** which is equal to one of the lattice vectors in magnitude and direction; the constant vector **b** is called the *Burgers vector* of the dislocation concerned. This property may be expressed as

$$\oint_L \mathrm{d}u_i = \oint \frac{\partial u_i}{\partial x_k}\mathrm{d}x_k = -b_i, \tag{27.1}$$

where the direction in which the contour is traversed and the chosen direction of the tangent vector $\boldsymbol{\tau}$ to the dislocation line are assumed to be related by the corkscrew rule† (Fig. 24). The dislocation line itself is a line of singularities of the deformation field.

It is evident that the Burgers vector **b** is necessarily constant along the dislocation line, and also that this line cannot simply terminate within the crystal: it must either reach the surface of the crystal at both ends or (as usually happens in actual cases) form a closed loop.

† The simple cases of edge and screw dislocations mentioned above correspond to straight lines D with $\boldsymbol{\tau} \perp \mathbf{b}$ and $\boldsymbol{\tau} \parallel \mathbf{b}$. We may also note that in the representation given by Fig. 22 edge dislocations with opposite directions of **b** differ in that the "extra" crystal half plane lies above or below the xz-plane; such dislocations are said to have opposite *signs*.

The condition (27.1) signifies, therefore, that in the presence of a dislocation the displacement vector is not a single-valued function of the co-ordinates, but receives a certain increment in a passage round the dislocation line.

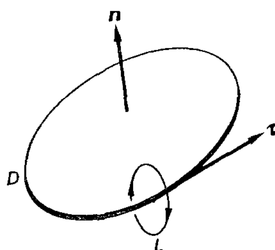


FIG. 24

Physically, of course, there is no ambiguity: the increment **b** denotes an additional displacement of the lattice points equal to a lattice vector, and this does not affect the lattice itself.

In the subsequent discussion it is convenient to use the notation

$$w_{ik} = \partial u_k / \partial x_i, \tag{27.2}$$

so that the condition (27.1) becomes

$$\oint_L w_{ik} \, \mathrm{d}x_i = -b_k. \tag{27.3}$$

The (unsymmetrical) tensor w_{ik} is called the *distortion tensor*. Its symmetrical part gives the ordinary strain tensor:

$$u_{ik} = \tfrac{1}{2}(w_{ik} + w_{ki}). \tag{27.4}$$

According to the foregoing discussion the tensors w_{ik} and u_{ik}, and therefore the stress tensor σ_{ik}, are single-valued functions of the co-ordinates, unlike the function $\mathbf{u}(\mathbf{r})$.

The condition (27.3) may also be written in a differential form. To do so, we transform the integral round the contour L into one over a surface S_L spanning this contour:†

$$\oint_L w_{mk} \, \mathrm{d}x_m = \int_{S_L} e_{ilm} \frac{\partial w_{mk}}{\partial x_l} \mathrm{d}f_i.$$

The constant vector b_k is written as an integral over the same surface by

† The transformation is made, according to STOKES' theorem, by replacing $\mathrm{d}x_m$ by the operator $\mathrm{d}f_i e_{ilm} \, \partial/\partial x_l$, where e_{ilm} is the antisymmetric unit tensor.

means of the two-dimensional delta function:

$$b_k = \int_{S_L} \tau_i b_k \delta(\boldsymbol{\xi}) \mathrm{d}f_i, \tag{27.5}$$

where $\boldsymbol{\xi}$ is the two-dimensional radius vector taken from the axis of the dislocation in the plane perpendicular to the vector $\boldsymbol{\tau}$ at the point considered. Since the contour L is arbitrary, the integrals can be equal only if the integrands are equal:

$$e_{ilm}\, \partial w_{mk}/\partial x_l = -\tau_i b_k \delta(\boldsymbol{\xi}). \tag{27.6}$$

This is the required differential form.†

The displacement field $\mathbf{u}(\mathbf{r})$ around the dislocation can be expressed in a general form if we know the GREEN's tensor $G_{ik}(\mathbf{r})$ of the equations of equilibrium of the anisotropic medium considered, i.e. the function which determines the displacement component u_i produced in an infinite medium by a unit force applied at the origin along the x_k-axis (see §8). This can easily be done by using the following formal device.

Instead of seeking many-valued solutions of the equations of equilibrium, we shall regard $\mathbf{u}(\mathbf{r})$ as a single-valued function, which undergoes a fixed discontinuity $\mathbf{b}$ on some arbitrarily chosen surface S_D spanning the dislocation loop D. Then the strain tensor formally defined by (27.4) will have a delta-function singularity on the "surface of discontinuity":

$$u_{ik}^{(S)} = \tfrac{1}{2}(n_i b_k + n_k b_i)\delta(\zeta), \tag{27.7}$$

where ζ is a co-ordinate measured from the surface S_D along the normal $\mathbf{n}$ (which is in the direction relative to $\boldsymbol{\tau}$ shown in Fig. 24).

Since there is no actual physical singularity in the space around the dislocation, the stress tensor σ_{ik} must, as already mentioned, be a single-valued and everywhere continuous function. The strain tensor (27.7), however, is formally related to a stress tensor $\sigma_{ik}^{(S)} = \lambda_{iklm}\, u_{lm}^{(S)}$, which also has a singularity on the surface S_D. In order to eliminate this we must define fictitious body forces distributed over the surface S_D with a certain density $f^{(S)}$. The equations of equilibrium in the presence of body forces are $\partial\sigma_{ik}/\partial x_k + f_i^{(S)} = 0$ (cf. (2.7)). Hence it is clear that we must put

$$f_i^{(S)} = -\frac{\partial \sigma_{ik}^{(S)}}{\partial x_k} = -\lambda_{iklm}\frac{\partial u_{lm}^{(S)}}{\partial x_k}. \tag{27.8}$$

Thus the problem of finding the many-valued function $\mathbf{u}(\mathbf{r})$ is equivalent to that of finding a single-valued but discontinuous function in the presence of

† To avoid misunderstanding it should be noted that on the dislocation line itself ($\xi \to 0$), which is a line of singularities, the representation of the w_{ik} as the derivatives (27.2) is no longer meaningful.

body forces given by formulae (27.7) and (27.8). We can now use the formula

$$u_i(\mathbf{r}) = \int G_{ij}(\mathbf{r}-\mathbf{r}')f_j^{(S)}(\mathbf{r}')\mathrm{d}V'.$$

We substitute (27.8) and integrate by parts; the integration with the delta function is then trivial, giving

$$u_i(\mathbf{r}) = -\lambda_{jklm}b_m \int\limits_{S_D} n_l \frac{\partial}{\partial x_k} G_{ij}(\mathbf{r}-\mathbf{r}')\mathrm{d}f'. \tag{27.9}$$

This solves the problem.†

The deformation (27.9) has its simplest form far from the closed dislocation loop. If we imagine the loop to be situated near the origin, then at distances r large compared with the linear dimensions of the loop we have

$$u_i(\mathbf{r}) = -\lambda_{jklm}d_{lm}\partial G_{ij}(\mathbf{r})/\partial x_k, \tag{27.10}$$

where

$$d_{ik} = S_i b_k,\ S_i = \int\limits_{S_D} n_i \mathrm{d}f = \tfrac{1}{2}e_{ikl}\oint\limits_{D} x_k \mathrm{d}x_l, \tag{27.11}$$

and e_{ikl} is the antisymmetric unit tensor. The axial vector $\mathbf{S}$ has components equal to the areas bounded by the projections of the loop D on planes perpendicular to the corresponding co-ordinate axes; the tensor d_{ik} may be called the *dislocation moment tensor*. The components of the tensor G_{ij} are first order homogeneous functions of the co-ordinates x, y, z (see §8, Problem). We therefore see from (27.10) that $u_i \sim 1/r^2$, and the corresponding stress field $\sigma_{ik} \sim 1/r^3$.

It is also easy to ascertain the way in which the elastic stresses vary with distance near a straight dislocation. In cylindrical polar co-ordinates z, r, ϕ (with the z-axis along the dislocation line) the deformation will depend only on r and ϕ. The integral (27.3) must, in particular, be unchanged by an arbitrary change in the size of any contour in the xy-plane which leaves the shape of the contour the same. It is clear that this can be true only if all the $w_{ik} \sim 1/r$. The tensor u_{ik}, and therefore the stresses σ_{ik}, will be proportional to the same power, $1/r$.‡

---

† The tensor G_{ij} for an anisotropic medium has been derived in the paper by I. M. Lifshitz and L. N. Rozentsveig quoted in §8, Problem. This tensor is in general very complicated. For a straight dislocation, which corresponds to a two-dimensional problem of elasticity theory, it may be simpler to solve the equations of equilibrium directly.

‡ Attention is drawn to a certain analogy between the elastic deformation field round a dislocation line and the magnetic field of constant line currents. The current is replaced by the Burgers vector, which must be constant along the dislocation line, like the current. Similar analogies will also be readily seen in the relations given below. However, quite apart from the entirely different nature of the two physical effects, these analogies are not far-reaching, because the tensor character of the corresponding quantities is different.

Although we have hitherto spoken only of dislocations, the formulae derived are applicable also to deformations caused by other kinds of defect in the crystal structure. Dislocations are linear defects; there exist also defects in which the regular structure is interrupted through a region near a given surface.† Such a defect can be macroscopically described as a surface of discontinuity on which the displacement vector **u** is discontinuous but the stresses σ_{ik} are continuous, by virtue of the equilibrium conditions. If the discontinuity **b** is the same everywhere on the surface, the resulting strain is just the same as that due to a dislocation along the edge of the surface. The only difference is that the vector **b** is not equal to a lattice vector. However, the position of the surface S_D discussed above is no longer arbitrary; it must coincide with the actual physical discontinuity. Such a surface of discontinuity involves a certain additional energy which may be described by means of an appropriate surface-tension coefficient.

PROBLEMS

PROBLEM 1. Derive the differential equations of equilibrium for a dislocation deformation in an isotropic medium, expressed in terms of the displacement vector.‡

SOLUTION. In terms of the stress tensor or strain tensor the equations of equilibrium have the usual form $\partial\sigma_{ik}/\partial x_k = 0$ or, substituting σ_{ik} from (5.11),

$$\frac{\partial u_{ik}}{\partial x_k}+\frac{\sigma}{1-2\sigma}\frac{\partial u_{ll}}{\partial x_i}=0. \tag{1}$$

To convert to the vector **u** we must use the differential condition (27.6). Multiplying (27.6) by e_{ikn} and summing over i and k, we obtain§

$$\frac{\partial w_{nk}}{\partial x_k}-\frac{\partial w_{kk}}{\partial x_n}=-(\boldsymbol{\tau}\times\mathbf{b})_n\delta(\boldsymbol{\xi}). \tag{2}$$

Writing (1) in the form

$$\tfrac{1}{2}\frac{\partial w_{ik}}{\partial x_k}+\tfrac{1}{2}\frac{\partial w_{ki}}{\partial x_k}+\frac{\sigma}{1-2\sigma}\frac{\partial w_{ll}}{\partial x_i}=0$$

and substituting (2), we find

$$\frac{\partial w_{ki}}{\partial x_k}+\frac{1}{1-2\sigma}\frac{\partial w_{ll}}{\partial x_i}=(\boldsymbol{\tau}\times\mathbf{b})_i\delta(\boldsymbol{\xi}).$$

Now changing to **u** in accordance with (27.2), we find the required equation for the multi-valued function $\mathbf{u}(\mathbf{r})$:

$$\triangle\mathbf{u}+\frac{1}{1-2\sigma}\,\mathbf{grad}\ \mathrm{div}\ \mathbf{u}=\boldsymbol{\tau}\times\mathbf{b}\delta(\boldsymbol{\xi}). \tag{3}$$

---

† A well-known example of a defect of this type is a narrow twinned layer in a crystal.

‡ The physical meaning of this and other problems relating to an isotropic medium is purely conventional, since actual dislocations by their nature occur only in crystals, i.e. in anisotropic media. Such problems have illustrative value, however.

§ Using also the formula $e_{ilm}e_{ikn} = \delta_{lk}\delta_{mn}-\delta_{ln}\delta_{mk}$.

PROBLEM 2. Determine the deformation near a straight screw dislocation in an isotropic medium.

SOLUTION. We take cylindrical polar co-ordinates z, r, ϕ, with the z-axis along the dislocation line; the Burgers vector is $b_x = b_y = 0$, $b_z = b$. It is evident from symmetry that the displacement $\mathbf{u}$ is parallel to the z-axis and is independent of the co-ordinate z. The equation of equilibrium (3), Problem 1, reduces to $\triangle u_z = 0$. The solution which satisfies the condition (27.1) is† $u_z = b\phi/2\pi$. The only non-zero components of the tensors u_{ik} and σ_{ik} are $u_{z\phi} = b/4\pi r$, $\sigma_{z\phi} = \mu b/2\pi r$, and the deformation is therefore a pure shear.

The free energy of the dislocation (per unit length) is given by the integral

$$F = \tfrac{1}{2}\int 2u_{z\phi}\sigma_{z\phi}\mathrm{d}V$$

$$= \frac{\mu b^2}{4\pi}\int\frac{\mathrm{d}r}{r},$$

which diverges logarithmically at both limits. As the lower limit we must take the order of magnitude of the interatomic distances ($\sim b$), at which the deformation is large and the macroscopic theory is inapplicable. The upper limit is determined by a dimension of the order of the length L of the dislocation. Then $F = (\mu b^2/4\pi)\log(L/b)$. The energy of the deformation in the "core" of the dislocation near its axis (in a region of cross-sectional area $\sim b^2$) can be estimated as $\sim \mu b^2$. When $\log(L/b) \gg 1$ this energy is small in comparison with that of the elastic deformation field.‡

PROBLEM 3. Determine the internal stresses in an anisotropic medium near a screw dislocation which is perpendicular to a plane of symmetry of the crystal.

SOLUTION. We take co-ordinates x, y, z so that the z-axis is along the dislocation line, and again write $b_z = b$. The vector $\mathbf{u}$ again has only the component $u_z = u(x, y)$. Since the xy-plane is a plane of symmetry, all the components of the tensor λ_{iklm} are zero which contain the suffix z an odd number of times. Thus only two components of the tensor σ_{ik} are non-zero

$$\sigma_{xz} = \lambda_{xzxz}\frac{\partial u}{\partial x} + \lambda_{xzyz}\frac{\partial u}{\partial y},$$

$$\sigma_{yz} = \lambda_{yzxz}\frac{\partial u}{\partial x} + \lambda_{yzyz}\frac{\partial u}{\partial y}.$$

We define a two-dimensional vector $\boldsymbol{\sigma}$ and a two-dimensional tensor $\lambda_{\alpha\beta}$: $\sigma_\alpha \equiv \sigma_{\alpha z}$, $\lambda_{\alpha\beta} \equiv \lambda_{\alpha z\beta z}$ ($\alpha = 1, 2$). Then $\sigma_\alpha = \lambda_{\alpha\beta}\partial u/\partial x_\beta$, and the equation of equilibrium becomes $\operatorname{div}\boldsymbol{\sigma} = 0$. The required solution of this equation must satisfy the condition (27.1): $\oint \mathbf{grad}\, u \cdot \mathrm{d}\mathbf{l} = -b$.

In this form, the problem is the same as that of finding the magnetic induction and magnetic field (represented by $\boldsymbol{\sigma}$ and **grad** u) in an anisotropic medium of magnetic permeability $\lambda_{\alpha\beta}$ near a straight current of strength $I = cb/4\pi$. Using the solution derived in electrodynamics, we obtain§

$$\sigma_{\alpha z} = -\frac{b}{2\pi}\frac{\lambda_{\alpha\beta}e_{\beta\gamma z}x_\gamma}{\sqrt{|\lambda|}.\lambda^{-1}_{\alpha'\beta'}x_{\alpha'}x_{\beta'}},$$

where $|\lambda|$ is the determinant of the tensor $\lambda_{\alpha\beta}$.

---

† In all the problems on straight dislocations we take the vector $\boldsymbol{\tau}$ in the negative z-direction.

‡ These estimates are general ones and are valid in order of magnitude for any dislocation (and not only for a screw dislocation).

It should be noted that in practice the values of $\log(L/b)$ are usually not very large, and the energy of the "core" is therefore a considerable fraction of the total energy of the dislocation.

§ See *Electrodynamics of Continuous Media*, §29, Problem 5.

PROBLEM 4. Determine the deformation near a straight edge dislocation in an isotropic medium.

SOLUTION. Let the z-axis be along the dislocation line, and the Burgers vector be $b_x = b$, $b_y = b_z = 0$. It is evident from the symmetry of the problem that the displacement vector lies in the xy-plane and is independent of z, so that the problem is a two-dimensional one. In the rest of this solution all vectors and vector operations are two-dimensional in the xy-plane.

We shall seek a solution of the equation

$$\triangle \mathbf{u}+\frac{1}{1-2\sigma}\,\mathbf{grad}\ \mathrm{div}\ \mathbf{u} = -b\mathbf{j}\delta(\mathbf{r})$$

(see Problem 1; $\mathbf{j}$ is a unit vector along the y-axis) in the form $\mathbf{u} = \mathbf{u}^{(0)}+\mathbf{w}$, where $\mathbf{u}^{(0)}$ is a vector with components $u^{(0)}{}_x = b\phi/2\pi$, $u^{(0)}{}_y = (b/2\pi)\log r$; these are the imaginary and real parts of $(b/2\pi)\log(x+iy)$, r and ϕ being polar co-ordinates in the xy-plane. This vector satisfies the condition (27.1). The problem therefore reduces to finding the single-valued function $\mathbf{w}$. Since, as is easily verified, $\mathrm{div}\,\mathbf{u}^{(0)} = 0$, $\triangle\mathbf{u}^{(0)} = b\mathbf{j}\delta(\mathbf{r})$, it follows that $\mathbf{w}$ satisfies the equation

$$\triangle \mathbf{w}+\frac{1}{1-2\sigma}\,\mathbf{grad}\ \mathrm{div}\ \mathbf{w} = -2b\mathbf{j}\delta(\mathbf{r}).$$

This is the equation of equilibrium under forces concentrated along the z-axis with volume density $Eb\mathbf{j}\delta(\mathbf{r})/2(1+\sigma)$; cf. §8, Problem, equation (1). By means of the GREEN's tensor found in that problem for an infinite medium, the calculation of $\mathbf{w}$ is reduced to that of the integral

$$\mathbf{w} = \frac{b}{8\pi(1-\sigma)}.2\int_0^\infty\left[\frac{(3-4\sigma)\mathbf{j}}{R}+\frac{\mathbf{r}y}{R^3}\right]\mathrm{d}z',$$

$$R = \sqrt{(r^2+z'^2)}.$$

The result is

$$u_x = \frac{b}{2\pi}\left\{\tan^{-1}\frac{y}{x}+\frac{1}{2(1-\sigma)}\frac{xy}{x^2+y^2}\right\},$$

$$u_y = -\frac{b}{2\pi}\left\{\frac{1-2\sigma}{2(1-\sigma)}\log\sqrt{(x^2+y^2)}+\frac{1}{2(1-\sigma)}\frac{x^2}{x^2+y^2}\right\}.$$

The stress tensor calculated from this has Cartesian components

$$\sigma_{xx} = -bD\frac{y(3x^2+y^2)}{(x^2+y^2)^2},$$

$$\sigma_{yy} = bD\frac{y(x^2-y^2)}{(x^2+y^2)^2},$$

$$\sigma_{xy} = bD\frac{x(x^2-y^2)}{(x^2+y^2)^2},$$

and polar components

$$\sigma_{rr} = \sigma_{\phi\phi} = -(bD/r)\sin\phi,$$

$$\sigma_{r\phi} = (bD/r)\cos\phi,$$

where $D = \mu/2\pi(1-\sigma)$.

Problem 5. An infinity of identical parallel straight edge dislocations in an isotropic medium lie in one plane perpendicular to their Burgers vectors and at equal distances h apart. Find the shear stresses due to such a "dislocation wall" at distances large compared with h.

Solution. Let the dislocations be in the yz-plane and parallel to the z-axis. According to the results of Problem 4, the total stress due to all the dislocations at the point (x, y) is given by the sum

$$\sigma_{xy}(x, y) = bDx\sum_{n=-\infty}^{\infty} \frac{x^2-(y-nh)^2}{[x^2+(y-nh)^2]^2}.$$

This may be written in the form

$$\sigma_{xy} = -bD\frac{\alpha}{h}\left[J(\alpha, \beta)+\alpha\frac{\partial J(\alpha, \beta)}{\partial\alpha}\right],$$

where

$$J(\alpha, \beta) = \sum_{n=-\infty}^{\infty}\frac{1}{\alpha^2+(\beta-n)^2},\quad \alpha = x/h,\ \beta = y/h.$$

According to Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi ikx}\mathrm{d}x,$$

we find

$$J(\alpha, \beta) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\alpha^2+\xi^2}+2\,\mathrm{re}\sum_{k=1}^{\infty} e^{2\pi ik\beta}\int_{-\infty}^{\infty}\frac{e^{2\pi ik\xi}\mathrm{d}\xi}{\alpha^2+\xi^2}$$

$$= \frac{\pi}{\alpha}+\frac{2\pi}{\alpha}\sum_{k=1}^{\infty} e^{-2\pi k\alpha}\cos 2\pi k\beta.$$

When $\alpha = x/h \gg 1$ only the first term need be retained in the sum over k, and the result is

$$\sigma_{xy} = 4\pi^2 D\frac{bx}{h^2}e^{-2\pi x/h}\cos(2\pi y/h).$$

Thus the stresses decrease exponentially away from the wall.

§28. The action of a stress field on a dislocation

Let us consider a dislocation loop D in a field of elastic stresses $\sigma_{ik}^{(e)}$ created by given external loads, and calculate the force on the loop in such a field.

According to the general rules, this must be done by finding the work δR done by internal stresses in an infinitesimal displacement of the loop D. If

δu_{ik} is the change in the strain tensor due to this displacement, we have from (3.1)†

$$\delta R = -\int \sigma_{ik}^{(e)} \delta u_{ik} \mathrm{d}V.$$

Since the distribution of the stresses $\sigma_{ik}^{(e)}$ is assumed independent of the position of the dislocation, we can take the difference symbol δ outside the integral. Using also the symmetry of the tensor $\sigma_{ik}^{(e)}$ and the equation of equilibrium $\partial \sigma_{ik}^{(e)}/\partial x_k = 0$, we can write

$$\begin{aligned} \delta R &= -\delta \int \sigma_{ik}^{(e)} u_{ik} \mathrm{d}V \\ &= -\delta \int \sigma_{ik}^{(e)} \frac{\partial u_k}{\partial x_i} \mathrm{d}V \\ &= -\delta \int \frac{\partial}{\partial x_i} (\sigma_{ik}^{(e)} u_k) \mathrm{d}V. \end{aligned} \tag{28.1}$$

As explained in §27, we shall regard the displacement $\mathbf{u}$ as a single-valued function having a discontinuity on some surface S_D spanning the line D. Then the volume integral in (28.1) can be transformed into an integral over a closed surface consisting of the upper and lower surfaces of the cut S_D, joined by a tubular lateral surface of infinitesimal width enclosing the line D. The values of the continuous quantities $\sigma_{ik}^{(e)}$ are the same on both surfaces, but the values of $\mathbf{u}$ differ by a given amount $\mathbf{b}$. We therefore obtain‡

$$\delta R = -b_k \delta \int_{S_D} \sigma_{ik}^{(e)} \mathrm{d}f_i. \tag{28.2}$$

Let each element of length $\mathrm{d}\mathbf{l}$ of the dislocation be displaced by an amount $\delta \mathbf{r}$. This displacement causes a change in the area of the surface S_D, the elementary change being $\delta \mathbf{f} = \delta \mathbf{r} \times \mathrm{d}\mathbf{l}$, i.e. $\delta f_i = e_{imn} \delta x_m \mathrm{d}l_n = e_{imn} \delta x_m \tau_n \mathrm{d}l$. The work (28.2) therefore becomes a line integral round the dislocation loop:

$$\delta R = -\oint_D b_k e_{imn} \sigma_{ki}^{(e)} \delta x_m \tau_n \mathrm{d}l,$$

where $\boldsymbol{\tau}$ is the tangent vector to D.

The coefficient of δx_m in the integrand is minus the force f_m on unit length

† To avoid misunderstanding we must emphasise that δu_{ik} in this formula is to be taken (in accordance with the sense of this quantity in (3.1)) as the total (geometrical) change in the deformation following an infinitesimal movement of the dislocation. It comprises both elastic and plastic (see §29) parts.

‡ The integral over the tubular lateral surface of radius ρ vanishes as $\rho \to 0$, since the u_k become infinite more slowly than $1/\rho$.

of the dislocation line. Thus

$$f_i = e_{ikl}\tau_k\sigma_{lm}^{(e)}b_m \tag{28.3}$$

(M. PEACH and J. S. KOEHLER 1950). We may note that the force **f** is perpendicular to the vector $\boldsymbol{\tau}$, i.e. to the dislocation line, and also to the vector $\sigma_{ik}^{(e)}b_k$.

The plane which is defined by the vectors $\boldsymbol{\tau}$ and **b** at each point of the dislocation is called the *slip plane* of the corresponding element of the dislocation; for every element this plane of course touches the slip surface of the whole dislocation, which is a cylindrical surface with generators parallel to the Burgers vector **b** of the dislocation. The distinctive physical property of the slip plane is that it is the only one in which a comparatively easy mechanical displacement of the dislocation is possible.† For this reason it is of interest to determine the force (28.3) on this plane.

Let $\boldsymbol{\kappa}$ be a vector normal to the dislocation line in the slip plane. Then the required force component ($f_\perp$, say) is $f_\perp = \kappa_i f_i = e_{ikl}\kappa_i\tau_k b_m\sigma_{lm}^{(e)}$, or

$$f_\perp = \nu_l\sigma_{lm}^{(e)}b_m, \tag{28.4}$$

where $\boldsymbol{\nu} = \boldsymbol{\kappa}\times\boldsymbol{\tau}$ is a vector normal to the slip plane. Since the vectors **b** and $\boldsymbol{\nu}$ are perpendicular, we see that the force $f_\perp$ is determined by only one component $\sigma_{lm}^{(e)}$ if two of the co-ordinate axes are taken along these vectors.

The total force acting on the whole dislocation loop is

$$F_i = e_{ikl}b_m\oint_D \sigma_{lm}^{(e)}\mathrm{d}x_k. \tag{28.5}$$

This is zero except for a non-uniform stress field; when $\sigma_{lm}^{(e)}$ = constant, the integral is $\oint \mathrm{d}x_k = 0$. If the stress field varies only slightly over the loop, we can write

$$F_i = e_{ikl}b_m\frac{\partial\sigma_{lm}^{(e)}}{\partial x_p}\oint_D x_p\mathrm{d}x_k,$$

the loop being regarded as situated near the origin. This force can be expressed in terms of the dislocation moment d_{kl} defined by (27.11):

$$F_i = d_{kl}\partial\sigma_{kl}^{(e)}/\partial x_i. \tag{28.6}$$

PROBLEMS

PROBLEM 1. Find the force of interaction between two parallel screw dislocations in an isotropic medium.

---

† This fact follows from the microscopic form of a dislocation defect. For example, to move the edge dislocation shown in Fig. 22 in its slip plane (the xz-plane) comparatively slight movements of the atoms are sufficient, which make crystal planes farther and farther from the yz plane (but still parallel to it) into "extra" planes.

The movement of the dislocation in other directions can occur only by diffusion processes. For example, the dislocation shown in Fig. 22 can move in the yz plane only when atoms leave the "extra" half-plane by diffusion. Such a process can be of practical importance only at fairly high temperatures.

SOLUTION. The force per unit length acting on one dislocation in the stress field due to the other dislocation is determined from formula (28.4), using the results of §27, Problem 2. It is a radial force of magnitude $f = \mu b_1 b_2 / 2\pi r$. Dislocations of like sign ($b_1 b_2 > 0$) repel, while those of unlike sign ($b_1 b_2 < 0$) attract.

PROBLEM 2. A straight screw dislocation lies parallel to the plane free surface of an isotropic medium. Find the force acting on the dislocation.

SOLUTION. Let the yz-plane be the surface of the body, and let the dislocation be parallel to the z-axis with co-ordinates $x = x_0$, $y = 0$.

The stress field which leaves the surface of the medium a free surface is described by the sum of the fields of the dislocation and its image in the yz-plane, considered to lie in an infinite medium:

$$\sigma_{xz} = \frac{\mu b}{2\pi}\left[\frac{y}{(x-x_0)^2+y^2} - \frac{y}{(x+x_0)^2+y^2}\right],$$

$$\sigma_{yz} = -\frac{\mu b}{2\pi}\left[\frac{x-x_0}{(x-x_0)^2+y^2} - \frac{x+x_0}{(x+x_0)^2+y^2}\right].$$

Such a field exerts a force on the dislocation considered which is equal to the attraction exerted by its image, i.e. the dislocation is attracted to the surface of the medium by a force $f = \mu b^2/4\pi x_0$.

PROBLEM 3. Find the force of interaction between two parallel edge dislocations in an isotropic medium which are in parallel slip planes.

SOLUTION. Let the slip planes be parallel to the xz-plane and let the z-axis be parallel to the dislocation lines; as in §27, Problem 4, we put $\tau_z = -1$, $b_x = b$. Then the force on unit length of the dislocation in the field of elastic stresses σ_{ik} has components $f_x = b\sigma_{xy}$, $f_y = -b\sigma_{xx}$. In the case considered, σ_{ik} is determined by the expressions derived in §27, Problem 4. If one dislocation is along the z-axis, it exerts on the other dislocation (passing through the point $(x, y, 0)$) a force whose polar components are $f_r = b_1 b_2 D/r$, $f_\phi = (b_1 b_2 D/r)\sin 2\phi$, $D = \mu/2\pi(1-\sigma)$. The component of this force in the slip plane is $f_x = (b_1 b_2 D/r)\cos\phi\cos 2\phi$, which is zero when $\phi = \frac{1}{2}\pi$ or $\frac{1}{4}\pi$. The former position corresponds to stable equilibrium when $b_1 b_2 > 0$, the latter when $b_1 b_2 < 0$.

§29. A continuous distribution of dislocations

If a crystal contains several dislocations at the same time which are at relatively short distances apart (although far apart compared with the lattice constant, of course), it is useful to treat them by means of an averaging process: we consider "physically infinitesimal" volume elements in the crystal with a large number of dislocation lines through each.

An equation which expressed a fundamental property of dislocation deformations can be formulated by a natural generalisation of equation (27.6). We define a tensor ρ_{ik} (the *dislocation density tensor*) such that its integral over a surface spanning any contour L is equal to the sum $\mathbf{b}$ of the Burgers vectors of all the dislocation lines embraced by the contour:

$$\int_{S_L} \rho_{ik}\, df_i = b_k. \tag{29.1}$$

The continuous functions ρ_{ik} describe the distribution of dislocations in the

crystal. This tensor now replaces the expression on the right of equation (27.6):

$$e_{ilm} \partial w_{mk} / \partial x_l = -\rho_{ik}. \tag{29.2}$$

This equation shows that the tensor ρ_{ik} must satisfy the condition

$$\partial \rho_{ik} / \partial x_i = 0; \tag{29.3}$$

for a single dislocation, this condition simply states that the Burgers vector is constant along the dislocation line.

When the dislocations are treated in this way, the tensor w_{ik} becomes a primary quantity describing the deformation and determining the strain tensor through (27.4). A displacement vector **u** related to w_{ik} by the definition (27.2) cannot exist; this is clear from the fact that with such a definition the left-hand side of equation (29.2) would be identically zero throughout the crystal.

So far we have assumed the dislocations to be at rest. Let us now see how a set of equations may be formulated so as to allow in principle elastic deformations and stresses in a medium where dislocations are moving in a given manner† (E. Kröner and G. Rieder 1956).

Equation (29.2) is independent of whether the dislocations are at rest or in motion. The tensor w_{ik} still determines the elastic deformation; its symmetrical part is the elastic strain tensor, which is related to the stress tensor in the usual way, by Hooke's law.

This equation, however, is now insufficient for a complete formulation of the problem. The full set of equations must also determine the velocity **v** of the points in the medium.

It must be borne in mind that the movement of dislocations causes not only a change in the elastic deformation but also a change in the shape of the crystal which does not involve stresses, i.e. a *plastic deformation*. The motion of dislocations is in fact a mechanism of plastic deformation. This is clearly illustrated by Fig. 25, where the passage of the edge dislocation from left to right causes the part of the crystal above the slip plane to be shifted to the right by one lattice period; since the lattice is then regular, the crystal remains unstressed. Unlike an elastic deformation, which is uniquely defined by the thermodynamic state of the body, a plastic deformation depends on the process which occurs. In considering dislocations at rest we have no need to distinguish elastic and plastic deformations, since we are concerned only with stresses which are independent of the previous history of the crystal.

Let **u** be the geometrical displacement vector of points in the medium, measured, say, from their position before the deformation process begins; its

---

† We shall not discuss here the problem of determining this motion itself from the forces applied to the body. The solution of such a problem requires a detailed study of the microscopic mechanism of the motion of dislocations and their retardation by various defects, which must take account of the conditions occurring in actual crystals.

time derivative $\dot{\mathbf{u}} = \mathbf{v}$. If the "total distortion" tensor $W_{ik} = \partial u_k/\partial x_i$ is formed from the vector $\mathbf{u}$, its "plastic part" $w_{ik}^{(\mathrm{pl})}$ is obtained by subtracting

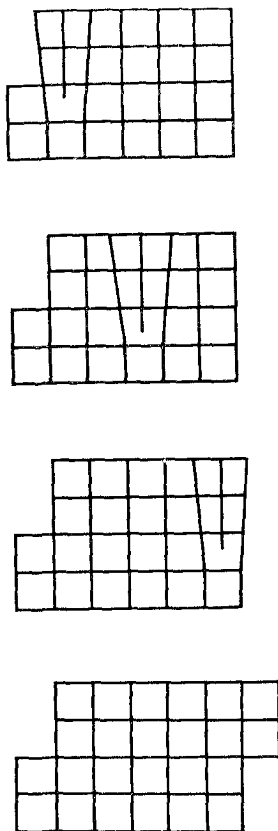


Fig. 25

from W_{ik} the "elastic distortion" tensor, which is the same as the tensor w_{ik} in (29.2). We use the notation

$$-j_{ik} = \partial w_{ik}^{(\mathrm{pl})}/\partial t; \tag{29.4}$$

the symmetrical part of j_{ik} gives the rate of variation of the plastic deformation tensor: the change in $u_{ik}^{(\mathrm{pl})}$ in an infinitesimal time interval δt is

$$\delta u_{ik}^{(\mathrm{pl})} = -\tfrac{1}{2}(j_{ik}+j_{ki})\delta t. \tag{29.5}$$

We may note, in particular, that, if a plastic deformation occurs without destroying the continuity of the body, the trace of the tensor j_{ik} is zero: a plastic deformation causes no extension or compression of the body (which would always involve the appearance of internal stresses), i.e. $u_{kk}^{(\mathrm{pl})} = 0$, and therefore $j_{kk} = -\partial u_{kk}^{(\mathrm{pl})}/\partial t = 0$.

Substituting in the definition (29.4) $w_{ik}^{(el)} = w_{ik} - w_{ik}^{(pl)}$, we can write it as

$$\frac{\partial w_{ik}^{(el)}}{\partial t} = \frac{\partial v_k}{\partial x_i} + j_{ik}, \tag{29.6}$$

n equation which relates the rates of change of the elastic and plastic deforma-ions. Here the j_{ik} must be regarded as given quantities which must satisfy onditions ensuring the compatibility of equations (29.6) and (29.2). These onditions are found by differentiating (29.2) with respect to time and sub-tituting (29.6), and are

$$\frac{\partial \rho_{ik}}{\partial t} + e_{ilm}\frac{\partial j_{mk}}{\partial x_l} = 0. \tag{29.7}$$

The complete set of equations is given by (29.2) and (29.6), together with the ynamical equations

$$\rho \dot{v}_i = \partial \sigma_{ik}/\partial x_k, \tag{29.8}$$

here $\sigma_{ik} = \lambda_{iklm}u_{lm} = \lambda_{iklm}w_{lm}$. The tensors ρ_{ik} and j_{ik} which appear in ıese equations are given functions of the co-ordinates (and time) which escribe the distribution and movement of the dislocations. These functions ıust satisfy the compatibility conditions of equations (29.2) with one another ıd with (29.6), which are given by (29.3) and (29.7).

The condition (29.7) may be regarded as a differential expression of the "law conservation of the Burgers vector" in the medium: integrating both sides this equation over a surface spanning some closed line L, defining by (29.1) e total Burgers vector **b** of the dislocations embraced by L, and using TOKES' theorem, we obtain

$$\frac{\mathrm{d}b_k}{\mathrm{d}t} = -\oint_L j_{ik}\mathrm{d}x_i. \tag{29.9}$$

ıe form of this equation shows that the integral on the right gives the "flux" the Burgers vector through the contour L per unit time, i.e. the Burgers ctor carried across L by moving dislocations. We may therefore call j_{ik} the *location flux density tensor.*

In particular, it is clear that for an isolated dislocation loop the tensor j_{ik} has e form

$$\begin{aligned} j_{ik} &= e_{ilm}\rho_{lk}V_m \\ &= e_{ilm}\tau_l V_m b_k \delta(\boldsymbol{\xi}), \end{aligned} \tag{29.10}$$

being given by (27.6), and **V** being the velocity of the dislocation line at a rticular point on it. The flux vector through the element d**l** of the contour s $j_{ik}\mathrm{d}l_i$ and is proportional to $\mathrm{d}\mathbf{l}\cdot\boldsymbol{\tau}\times\mathbf{V} = \mathbf{V}\cdot\mathrm{d}\mathbf{l}\times\boldsymbol{\tau}$, i.e. the component of **V** a direction perpendicular to both d**l** and $\boldsymbol{\tau}$; from geometrical considerations

it is evident that this is correct, since only that velocity component causes the dislocation to intersect the element $d\mathbf{l}$.

We may note that the trace of the tensor (29.10) is proportional to the component of the velocity of the dislocation along the normal to its slip plane. It has been mentioned above that the absence of any inelastic change in density of the medium is ensured by the condition $j_{ii} = 0$. We see that for an individual dislocation this condition signifies motion in the slip plane, in accordance with the previous discussion of the physical nature of the movement of dislocations; see the last footnote to §28.

Finally, let us consider the case where dislocation loops are distributed in the crystal in such a way that their total Burgers vector (denoted by $\mathbf{B}$) is zero.†

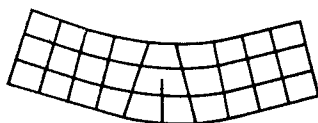


FIG. 26

This condition signifies that integration over any cross-section of the body gives

$$\int \rho_{ik} df_i = 0. \tag{29.11}$$

From this it follows that the dislocation density in this case can be written as

$$\rho_{ik} = e_{ilm} \partial P_{mk} / \partial x_l \tag{29.12}$$

(F. KROUPA 1962); then the integral (29.11) becomes an integral along a contour outside the body, and is zero. It may also be noted that the expression (29.12) necessarily satisfies the condition (29.3).

It is easy to see that the tensor P_{ik} thus defined represents the dislocation moment density in the deformed crystal, and may therefore be called the "dislocation polarisation": the total dislocation moment D_{ik} of the crystal is, by definition,

$$D_{ik} = \sum S_i b_k = \tfrac{1}{2} e_{ilm} \sum b_k \oint_D x_l dx_m$$

$$= \tfrac{1}{2} \int e_{ilm} x_l \rho_{mk} dV,$$

† The presence of a dislocation involves a certain bending of the crystal, as shown diagrammatically in Fig. 26 (greatly exaggerated). The condition $\mathbf{B} = 0$ means that there is no macroscopic bending of the crystal as a whole.

where the summation is over all dislocation loops and the integration is over the whole volume of the crystal. Substituting (29.12), we obtain

$$D_{ik} = \tfrac{1}{2}\int e_{ilm}e_{mpq}x_l \frac{\partial P_{qk}}{\partial x_p}\,dV$$

$$= \tfrac{1}{2}\int x_m\left(\frac{\partial P_{mk}}{\partial x_i} - \frac{\partial P_{ik}}{\partial x_m}\right)dV$$

and, after integrating by parts in each term,

$$D_{ik} = \int P_{ik}\,dV. \qquad (29.13)$$

The dislocation flux density is given in terms of the same tensor P_{ik} by

$$j_{ik} = -\partial P_{ik}/\partial t. \qquad (29.14)$$

This is easily seen, for example, by calculating the integral $\int j_{ik}\,dV$ over an arbitrary part of the volume of the body, using the expression (29.10), to give a sum over all dislocation loops within that volume. We may note that the expression (29.14) together with (29.12) automatically satisfies the condition (29.7).

A comparison of (29.14) with (29.4) shows that $\delta w_{ik}^{(pl)} = \delta P_{ik}$. If we agree to regard the plastic deformation as absent in the state with $P_{ik} = 0$, then $w_{ik}^{(pl)} = P_{ik}$,† and

$$w_{ik} = W_{ik} - w_{ik}^{(pl)} = \partial u_k/\partial x_i - P_{ik}, \qquad (29.15)$$

where u_k is again the vector of the total geometrical displacement from the position in the undeformed state. Equation (29.6) is then satisfied identically, and the dynamical equation (29.8) becomes

$$\rho\ddot{u}_i - \lambda_{iklm}\partial^2 u_m/\partial x_k\partial x_l = -\lambda_{iklm}\partial P_{lm}/\partial x_k. \qquad (29.16)$$

Thus the determination of the elastic deformation due to moving dislocations with $\mathbf{B} = 0$ reduces to a problem of ordinary elasticity theory with body forces distributed in the crystal with density $-\lambda_{iklm}\partial P_{lm}/\partial x_k$ (A. M. Kosevich 1963).

§30. Distribution of interacting dislocations

Let us consider a large number of similar straight dislocations lying parallel in the same slip plane, and derive an equation to determine their equilibrium distribution. Let the z-axis be parallel to the dislocations, and the xz-plane be the slip plane.

---

† It is assumed that the entire deformation process occurs with $\mathbf{B} = 0$. This point must be emphasised, since there is a fundamental difference between the tensors P_{ik} and $w_{ik}^{(pl)}$; whereas P_{ik} is a function of the state of the body, the tensor $w_{ik}^{(pl)}$ is not, but depends on the process which has brought the body into that state.

We shall suppose for definiteness that the Burgers vectors of the dislocations are in the x-direction. Then the force in the slip plane on unit length of a dislocation is $b\sigma_{xy}$, where σ_{xy} is the stress at the position of the dislocation.

The stresses created by one straight dislocation (and acting on another dislocation) decrease inversely as the distance from it. The stress at a point x due to a dislocation at a point x' is therefore $bD/(x-x')$, where D is a constant of the order of the elastic moduli of the crystal. It may be shown that this constant D is positive, i.e. two like dislocations in the same slip plane repel each other.†

Let $\rho(x)$ be the line density of dislocations on a segment (a_1, a_2) of the x-axis; $\rho(x)\mathrm{d}x$ is the sum of the Burgers vectors of dislocations passing through points in the interval $\mathrm{d}x$. Then the total stress at a point x on the x-axis due to all the dislocations is given by the integral

$$\sigma_{xy}(x) = -D\int_{a_1}^{a_2}\frac{\rho(\xi)\mathrm{d}\xi}{\xi-x}. \tag{30.1}$$

For points in the segment (a_1, a_2) this integral must be taken as a principal value in order to exclude the physically meaningless action of a dislocation on itself.

If the crystal is also subjected to a two-dimensional stress field $\sigma_{xy}{}^{(e)}(x, y)$ in the xy-plane, caused by given external loads, each dislocation will be subjected to a force $b(\sigma_{xy}+p(x))$, where for brevity $p(x)$ denotes $\sigma_{xy}{}^{(e)}(x, 0)$. The condition of equilibrium is that this force should be zero: $\sigma_{xy}+p = 0$, i.e.

$$P\int_{a_1}^{a_2}\frac{\rho(\xi)\mathrm{d}\xi}{\xi-x} = \frac{p(x)}{D} \equiv \omega(x), \tag{30.2}$$

where P denotes, as usual, the principal value. This is an integral equation to determine the equilibrium distribution $\rho(x)$. It is a singular equation with a Cauchy kernel.

The solution of such an equation is equivalent to a problem in the theory of functions of a complex variable which may be formulated as follows.

Let $\Omega(z)$ denote a function defined throughout the complex z-plane (cut from a_1 to a_2) as the integral

$$\Omega(z) = \int_{a_1}^{a_2}\frac{\rho(\xi)\mathrm{d}\xi}{\xi-z}. \tag{30.3}$$

Let $\Omega^+(x)$ and $\Omega^-(x)$ denote the limiting values of $\Omega(z)$ on the upper and lower edges of the cut. They are equal to similar integrals along the segment (a_1, a_2)

† For an isotropic medium this has been proved in §28, Problem 1.

with an indentation in the form of an infinitesimal semicircle below or above the point $z = x$ respectively, i.e.

$$\Omega^{\pm}(x) = P\int\limits_{a_1}^{a_2} \frac{\rho(\xi)\mathrm{d}\xi}{\xi - x} \pm i\pi\rho(x). \tag{30.4}$$

If $\rho(\xi)$ satisfies equation (30.2), the principal value of the integral is $\omega(x)$, and we therefore have

$$\Omega^{+}(x) + \Omega^{-}(x) = 2\omega(x), \tag{30.5}$$

$$\Omega^{+}(x) - \Omega^{-}(x) = 2i\pi\rho(x). \tag{30.6}$$

Thus the problem of solving equation (30.2) is equivalent to that of finding an analytic function $\Omega(z)$ with the property (30.5); $\rho(x)$ is then given by (30.6). The physical conditions of the problem in question also require that $\Omega(\infty) = 0$; this follows because far from the dislocations ($x \to \pm\infty$) the stresses σ_{xy} must be zero (by the definition (30.3), $\sigma_{xy}(x) = -D\Omega(x)$ outside the segment (a_1, a_2)).

Let us first consider the case where there are no external stresses ($p(x) = 0$), and the dislocations are constrained by some obstacles (lattice defects) at the ends of the segment (a_1, a_2). When $\omega(x) = 0$ we have from (30.5) $\Omega^{+}(x) = -\Omega^{-}(x)$, i.e. the function $\Omega(z)$ must change sign in a passage round each of the points a_1, a_2. This condition is satisfied by any function of the form

$$\Omega(z) = \frac{P(z)}{\sqrt{[(a_2 - z)(z - a_1)]}}, \tag{30.7}$$

where $P(z)$ is a polynomial. The condition $\Omega(\infty) = 0$ means that we must take $P(z) = 1$ (apart from a constant coefficient), so that

$$\Omega(z) = \frac{1}{\sqrt{[(a_2 - z)(z - a_1)]}}. \tag{30.8}$$

The required function $\rho(x)$ will, according to (30.6), have the same form. The coefficient is determined from the condition

$$\int\limits_{a_1}^{a_2} \rho(\xi)\mathrm{d}\xi = B, \tag{30.9}$$

where B is the sum of the Burgers vectors of all the dislocations, and so we have

$$\rho(x) = \frac{B}{\pi\sqrt{[(a_2 - x)(x - a_1)]}}. \tag{30.10}$$

We see that the dislocations pile up towards the obstacles at the ends of the

segment, with density inversely proportional to the square root of the distance from the obstacle. The stress outside the segment (a_1, a_2) increases in the same manner as the ends of the segment are approached, e.g. for $x > a_2$

$$\sigma_{xy} \cong \frac{BD}{\sqrt{[(x-a_2)(a_2-a_1)]}}.$$

In other words, the concentration of dislocations at the boundary leads to a stress concentration beyond the boundary.

Let us now suppose that under the same conditions (obstacles at the fixed ends of the segment) there is also an external stress field $p(x)$. Let $\Omega_0(z)$ denote a function of the form (30.7), and let us rewrite equation (30.5) divided by $\Omega_0{}^+ = -\Omega_0{}^-$ as

$$\frac{\Omega^+(x)}{\Omega_0{}^+(x)} - \frac{\Omega^-(x)}{\Omega_0{}^-(x)} = \frac{2\omega(x)}{\Omega_0{}^+(x)}.$$

A comparison of this with (30.6) shows that

$$\frac{\Omega(z)}{\Omega_0(z)} = \frac{1}{i\pi}\int_{a_1}^{a_2} \frac{\omega(\xi)}{\Omega_0{}^+(\xi)} \frac{d\xi}{\xi - z} + i\pi P(z), \tag{30.11}$$

where $P(z)$ is a polynomial. A solution which satisfies the condition $\Omega(\infty) = 0$ is obtained by taking as $\Omega_0(z)$ the function (30.8) and putting $P(z) = C$, a constant. The required function $\rho(x)$ is hence found by means of (30.6), and the result is

$$\rho(x) = -\frac{1}{\pi^2\sqrt{[(a_2-x)(x-a_1)]}} P\int_{a_1}^{a_2} \omega(\xi)\sqrt{[(a_2-\xi)(\xi-a_1)]}\frac{d\xi}{\xi - x} + \frac{C}{\sqrt{[(a_2-x)(x-a_1)]}}. \tag{30.12}$$

The constant C is determined by the condition (30.9). Here also $\rho(x)$ increases as $(a_2-x)^{-1/2}$ when $x \to a_2$ (and similarly when $x \to a_1$), and a similar concentration of stresses occurs on the other side of the boundary.

If there is an obstacle only on one side (at a_2, say) the required solution must satisfy the condition of finite stress for all $x < a_2$, including the point $x = a_1$; the position of the latter point is not known beforehand and must be determined by solving the problem. With respect to $\Omega(z)$ this means that $\Omega(a_1)$ must be finite. Such a function (satisfying also the condition $\Omega(\infty) = 0$) is obtained from the same formula (30.11) by taking for $\Omega_0(z)$ the function

$\surd[(z-a_1)/(a_2-z)]$, which is also of the form (30.7), and putting $P(z) = 0$ in (30.11). The result is

$$\rho(x) = -\frac{1}{\pi^2}\sqrt{\frac{x-a_1}{a_2-x}}\,P\int\limits_{a_1}^{a_2}\sqrt{\frac{a_2-\xi}{\xi-a_1}}\,\frac{\omega(\xi)\mathrm{d}\xi}{\xi-x}. \tag{30.13}$$

When $x\to a_1$, $\rho(x)$ tends to zero as $\surd(x-a_1)$. The total stress $\sigma_{xy}(x)+p(x)$ tends to zero according to a similar law on the other side of the point a_1.

Finally, let there be no obstacle at either end of the segment, and let the dislocations be constrained only by external stresses $p(x)$. The corresponding $\Omega(z)$ is obtained by putting in (30.11) $\Omega_0(z) = \surd[(a_2-z)(z-a_1)]$, $P(z) = 0$. The condition $\Omega(\infty) = 0$, however, here requires the fulfilment of a further condition: taking the limit as $z\to\infty$ in (30.11), we find

$$\int\limits_{a_1}^{a_2}\frac{\omega(\xi)\mathrm{d}\xi}{\surd[(a_2-\xi)(\xi-a_1)]} = 0. \tag{30.14}$$

The function $\rho(x)$ is given by

$$\rho(x) = -\frac{1}{\pi^2}\surd[(a_2-x)(x-a_1)]P\int\limits_{a_1}^{a_2}\frac{\omega(\xi)}{\surd[(a_2-\xi)(\xi-a_1)]}\,\frac{\mathrm{d}\xi}{\xi-x}, \tag{30.15}$$

the co-ordinates a_1 and a_2 of the ends of the segment being determined by the conditions (30.9) and (30.14).

PROBLEM

Find the distribution of dislocations in a uniform stress field $p(x) = -p_0$ over a segment with obstacles at one or both ends.

SOLUTION. When there is an obstacle at one end (a_2) the calculation of the integral (30.13) gives

$$\rho(x) = \frac{p_0}{\pi D}\sqrt{\frac{x-a_1}{a_2-x}}.$$

The condition (30.9) determines the length of the segment occupied by dislocations, $a_2-a_1 = 2BD/P_0$. Beyond the obstacle there is a concentration of stresses near it according to

$$\sigma_{xy} \cong p_0\sqrt{\frac{a_2-a_1}{x-a_2}}.$$

For a segment of length $2L$ bounded by two obstacles we take the origin of x at the midpoint and obtain from (30.12)

$$\rho(x) = \frac{1}{\pi\surd(L^2-x^2)}\left(\frac{p_0}{D}x+B\right).$$

§31. Equilibrium of a crack in an elastic medium

The problem of the equilibrium of a crack is somewhat distinctive among the problems of elasticity theory. From the point of view of that theory, a crack is a cavity in an elastic medium, which exists when internal stresses are present in the medium and closes up when the load is removed. The shape and size of the crack depend considerably on the stresses acting on it. The mathematical feature of the problem is therefore that the boundary conditions are given on a surface which is initially unknown and must itself be determined in solving the problem.†

Let us consider a crack in an isotropic medium, of infinite length and uniform in the z-direction and in a plane stress field $\sigma_{ik}^{(e)}(x, y)$; this is a two-dimensional problem of elasticity theory. We shall suppose that the stresses are symmetrical about the centre of the cross-section of the crack. Then the outline of the cross-section will also be symmetrical (Fig. 27). Let its length be $2L$ and its variable width $h(x)$; since the crack is symmetrical, $h(-x) = h(x)$

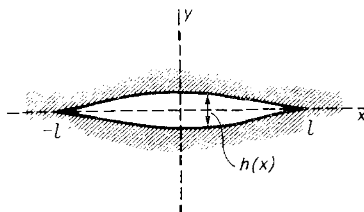


Fig. 27

We shall assume the crack to be thin ($h \ll L$). Then the boundary conditions on its surface can be applied to the corresponding segment of the x-axis. Thus the crack is regarded as a line of discontinuity (in the xy-plane) on which the normal component of the displacement $u_y = \pm \frac{1}{2}h$ is discontinuous.

Instead of $h(x)$ we define a new unknown function $\rho(x)$ by the formulae

$$h(x) = \int\limits_x^L \rho(x)\mathrm{d}x, \qquad \rho(-x) = -\rho(x). \tag{31.1}$$

The function $\rho(x)$ may be conveniently, though purely formally, interpreted as a density of straight dislocations lying in the z-direction and continuously distributed along the x-axis, with their Burgers vectors in the y-direction.‡ It has been shown in §27 that a dislocation line may be regarded as the edge of a surface of discontinuity on which the displacement $\mathbf{u}$ has a discontinuity $\mathbf{b}$. In the form (31.1) the discontinuity h of the normal displacement at the point

† The quantitative theory of cracks discussed here is due to G. I. Barenblatt (1959).

‡ It is for this reason that the theory of cracks is described here in the chapter on dislocations, although physically the phenomena are quite different.

x is regarded as the sum of the Burgers vectors of all the dislocations lying to the right of that point; the equation $\rho(-x) = -\rho(x)$ signifies that the dislocations to the right and to the left of the point $x = 0$ have opposite signs.

By means of this representation we can write down immediately an expression for the normal stresses (σ_{yy}) on the x-axis. These consist of the stresses $\sigma_{yy}{}^{(e)}(x, 0)$ resulting from the external loads (which for brevity we denote by $p(x)$) and the stresses $\sigma_{yy}{}^{(\mathrm{cr})}(x)$ due to the deformation caused by the crack. Regarding the latter stresses as being due to dislocations distributed over the segment $(-L, L)$, we obtain (similarly to (30.1))

$$\sigma_{yy}{}^{(\mathrm{cr})}(x) = -D\int\limits_{-L}^{L} \frac{\rho(\xi)\mathrm{d}\xi}{\xi - x}; \qquad (31.2)$$

for points in the segment $(-L, L)$ itself, the integral must be taken as a principal value. For an isotropic medium,

$$D = \frac{\mu}{2\pi(1-\sigma)} = \frac{E}{4\pi(1-\sigma^2)}; \qquad (31.3)$$

see §28, Problem 3. The stresses σ_{xy} due to such dislocations in an isotropic medium are zero on the x-axis.

The boundary condition on the free surface of the crack, applied (as already mentioned) to the corresponding segment of the x-axis, requires that the normal stresses $\sigma_{yy} = \sigma_{yy}{}^{(\mathrm{cr})} + p(x)$ should be zero. This condition, however, needs to be made more precise, for the following reason.

Let us make the assumption (which will be confirmed by the result) that the edges of the crack join smoothly near its ends, so that the surfaces approach very closely. Then it is necessary to take into account the forces of molecular attraction between the surfaces; the action of these forces extends to a distance r_0 large compared with interatomic distances. These forces will be of importance in a narrow region near the end of the crack where $h \sim r_0$; the length of this region will be denoted by d in order of magnitude, and will be estimated later.

Let G be the force of molecular cohesion per unit area of the crack; it depends on the distance h between the surfaces.† When these forces are taken into account, the boundary condition becomes

$$\sigma_{yy}{}^{(\mathrm{cr})} + p(x) - G = 0. \qquad (31.4)$$

It is reasonable to suppose that the shape of the crack near its end is determined by the nature of the cohesion forces and does not depend on the external loads applied to the body. Then, in finding the shape of the main part of the crack from the external forces $p(x)$, the quantity G becomes a given function $G(x)$ independent of $p(x)$ (over the region d, outside which it is unimportant).

† In the macroscopic theory, the function $G(x)$ is to be regarded as increasing smoothly, as $L - x$ decreases, up to a maximum value at the end of the crack.

Substituting $\sigma_{yy}^{(\mathrm{cr})}$ from (31.2) in (31.4), we thus obtain the following integral equation for $\rho(x)$:

$$P\int_{-L}^{L}\frac{\rho(\xi)\mathrm{d}\xi}{\xi-x}=\frac{1}{D}p(x)-\frac{1}{D}g(x)\equiv\omega(x). \tag{31.5}$$

Since the ends of the crack are assumed not fixed, the stresses must remain finite there. This means that, in solving the integral equation (31.5), we now have the last of the cases discussed in §30, for which the solution is given by (30.15). With the origin at the midpoint of the segment $(-L, L)$ this formula becomes

$$\rho(x)=-\frac{1}{\pi^2}\sqrt{(L^2-x^2)}P\int_{-L}^{L}\frac{\omega(\xi)}{\sqrt{(L^2-\xi^2)}}\frac{\mathrm{d}\xi}{\xi-x}. \tag{31.6}$$

The condition (30.14) must be satisfied, which in this case gives

$$\int_{0}^{L}\frac{p(x)\mathrm{d}x}{\sqrt{(L^2-x^2)}}-\int_{0}^{L}\frac{G(x)\mathrm{d}x}{\sqrt{(L^2-x^2)}}=0 \tag{31.7}$$

(where the integrals from $-L$ to L have been replaced by integrals from 0 to L, using the symmetry of the problem). Since $G(x)$ is zero except in the range $L-x\sim d$, in the second integral we can put $L^2-x^2\cong 2L(L-x)$; the condition (31.7) then becomes

$$\int_{0}^{L}\frac{p(x)\mathrm{d}x}{\sqrt{(L^2-x^2)}}=\frac{M}{\sqrt{(2L)}}, \tag{31.8}$$

where M denotes the constant

$$M=\int_{0}^{d}\frac{G(\xi)\mathrm{d}\xi}{\sqrt{\xi}}, \tag{31.9}$$

which depends on the medium concerned. This constant can be expressed in terms of the ordinary macroscopic properties of the body, its elastic moduli and surface tension α; as will be shown later, the relation is

$$M=\sqrt{[\pi\alpha E/(1-\sigma^2)]}. \tag{31.10}$$

The equation (31.8) determines the length $2L$ of the crack from the given stress distribution $p(x)$. For example, for a crack widened by concentrated

forces f applied to the midpoints of the sides ($p(x) = f\delta(x)$) we find

$$2L = f^2/M^2 = f^2(1-\sigma^2)/\pi\alpha E. \qquad (31.11)$$

It must be remembered, however, that stable equilibrium of a crack is not possible for every distribution $p(x)$. For instance, with uniform widening stresses ($p(x) = \text{constant} \equiv p_0$) (31.8) gives

$$2L = 4M^2/\pi^2 p_0^2 = 4\alpha E/\pi(1-\sigma^2)p_0^2. \qquad (31.12)$$

This inverse relation (L decreasing when p_0 increases) shows that the state is unstable. The value of L determined by (31.12) corresponds to unstable equilibrium and gives the "critical" crack length: longer cracks grow spontaneously, but shorter ones close up, a result first derived by A. A. GRIFFITH (1920).

Let us now return to the consideration of the shape of the crack. When $L-x \sim d$, the region $L-\xi \sim d$ is the most important in the integral in (31.6). The integral can then be replaced by its limiting value as $x \to L$; the result is $\rho = \text{constant} \times \sqrt{(L-x)}$, whence†

$$h(x) = \text{constant} \times (L-x)^{3/2} \qquad (L-x \sim d). \qquad (31.13)$$

We see that over the terminal region d the two sides of the crack in fact join smoothly. The value of the coefficient in (31.13) depends on the properties of the cohesion forces and can not be expressed in terms of the ordinary macroscopic parameters.‡

For the part farther from the end, where $d \ll L-x \ll L$, the region $L-\xi \sim d$ is again the most important in the integral in (31.6), and $\omega(\xi) \cong G(\xi)/D$. In addition to putting $L^2-x^2 \cong 2L(L-x)$, $L^2-\xi^2 \cong 2L(L-\xi)$, we can here replace $\xi - x$ by $L-x$, obtaining $\rho = M/\pi^2 D\sqrt{(L-x)}$, where M is the same constant as in (31.9), (31.10). Hence

$$h(x) = 2M\sqrt{(L-x)}/\pi^2 D \qquad (d \ll L-x \ll L). \qquad (31.14)$$

Thus the end of the crack has a shape independent of the applied forces (and therefore of the length of the crack) throughout the range $L-x \ll L$: when $L-x \gg d$ the shape is given by (31.14), and when $L-x \sim d$ it has an infinitely

† In order to proceed to the limit we must first divide the integral in (31.6) into two integrals with numerators $\omega(\xi) - \omega(L)$ and $\omega(L)$; the second integral makes no contribution to the limiting value.

‡ An estimate of the coefficient in (31.13) gives a value of the order of $\sqrt{a/d}$, where a is the dimensions of an atom (using $\alpha \sim aE$, $M \sim E\sqrt{a}$). An estimate of the length d is obtained from the condition $h(d) \sim r_0$, whence $d \sim r_0^2/a \sim r_0$. It should be mentioned, however, that in practice the required inequalities are satisfied only by a small margin, so that the resulting shape of the terminal projection of the crack is not to be taken as exact.

sharp projection (31.13) (Fig. 28). The shape of the remainder of the crack does depend on the applied forces.

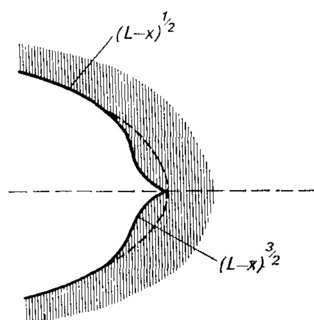


FIG. 28

If we ignore details, of the order of the radius of the action of the cohesion forces, the crack therefore has a smooth outline with ends rounded according to the parabolas (31.14), and this shape is entirely determined by the applied forces and the ordinary macroscopic parameters. The small ($\sim d$) terminal projections which actually occur are of fundamental significance, however, since they ensure that the stresses remain finite at the ends of the crack.

The stresses caused by the crack on the continuation of the x-axis are given by formula (31.2). At distances $x-L$ such that $d \ll x-L \ll L$, we have†

$$\sigma_{yy} \cong \sigma_{yy}^{(\mathrm{cr})} \cong M/\pi\sqrt{(x-L)}. \tag{31.15}$$

The increase in the stresses as the edge of the crack is approached continues according to this law up to distances $x-L \sim d$, and σ_{yy} then drops to zero at the point $x = L$.

It remains to derive the formula (31.10) already given above, which relates the constant M to the ordinary macroscopic quantities. To do this, we write down the condition for the total free energy to be a minimum by equating to zero its variation under a change in the length L.

Firstly, when the length of the crack increases by δL the surface energy at its two free surfaces increases by $\delta F_{\mathrm{surf}} = 2\alpha\delta L$. Secondly, the "opening" of the crack end reduces the elastic energy F_{el} by $\frac{1}{2}\int \sigma_{yy}(x)\eta(x)\mathrm{d}x$, where $\eta(x)$ is the difference in width between the displaced and undisplaced crack shapes. Since the shape of the crack end is independent of its length, $\eta(x) = h(x-\delta L) - h(x)$. The stress $\sigma_{yy} = 0$ for $x<L$, and $h(x) = 0$ for $x>L$. Hence

$$\delta F_{\mathrm{el}} = -\tfrac{1}{2}\int_{L}^{L+\delta L} \sigma_{yy}(x)h(x-\delta L)\mathrm{d}x.$$

† The integral is easily calculated directly, but it is not necessary to do this if we use the relation between the functions $\rho(x)$ for $x < L$ and $\sigma_{yy}^{(\mathrm{cr})}$ for $x > L$, which is evident from the results of §30.

Substituting (31.14) and (31.15), we find

$$\delta F_{\text{el}} = -\frac{M^2}{\pi^3 D}\int\limits_{L}^{L+\delta L}\sqrt{\frac{L+\delta L-x}{x-L}}\,\mathrm{d}x$$

$$= -\frac{M^2}{\pi^3 D}\int\limits_{0}^{\delta L}\frac{\sqrt{y}\,\mathrm{d}y}{\sqrt{(\delta L-y)}}$$

$$= -\frac{M^2}{2\pi^2 D}\delta L.$$

Finally, the condition $\delta F_{\text{surf}}+\delta F_{\text{el}} = 0$ gives the relation $M^2 = 4\pi^2\alpha D$, and hence we have (31.10).†

† It may be noted that the theory described above, including the relation (31.10), is in fact applicable as it stands only to ideally brittle bodies, i.e. those which remain linearly elastic up to fracture, such as glass and fused quartz. In bodies which exhibit plasticity the formation of the crack may be accompanied by plastic deformation at its ends.

CHAPTER V

THERMAL CONDUCTION AND VISCOSITY IN SOLIDS

§32. The equation of thermal conduction in solids

NON-UNIFORM heating of a solid does not cause convection as it generally does in fluids. Hence the transfer of heat is effected in solids by thermal conduction alone. The processes of thermal conduction in solids are therefore described by somewhat simpler equations than those for fluids, where they are complicated by convection.

The equation of thermal conduction in a solid can be derived immediately from the law of conservation of energy in the form of an "equation of continuity for heat". The amount of heat absorbed per unit time in unit volume of the body is $T\partial S/\partial t$, where S is the entropy per unit volume. This must be put equal to $-\operatorname{div}\mathbf{q}$, where $\mathbf{q}$ is the heat flux density. This flux can almost always be written as $-\kappa\,\mathbf{grad}\,T$, i.e. it is proportional to the temperature gradient (κ being the thermal conductivity). Thus

$$T\partial S/\partial t = \operatorname{div}(\kappa\,\mathbf{grad}\,T). \qquad (32.1)$$

According to formula (6.4), the entropy can be written as

$$S = S_0(T) + K\alpha u_{ii},$$

where α is the thermal expansion coefficient and S_0 the entropy in the undeformed state. We shall suppose that, as usually happens, the temperature differences in the body are so small that quantities such as κ, α, etc. may be regarded as constants. Then equation (32.1), after substitution of the above expression for S, becomes

$$T\frac{\partial S_0}{\partial t} + \alpha KT\frac{\partial u_{ii}}{\partial t} = \kappa \triangle T.$$

According to a well-known formula of thermodynamics, we have

$$C_p - C_v = K\alpha^2 T,$$

whence

$$\alpha KT = (C_p - C_v)/\alpha.$$

The time derivative of S_0 can be written as $(\partial S_0/\partial T)\cdot(\partial T/\partial t)$, where the derivative $\partial S_0/\partial T$ is taken for $u_{ll} = \operatorname{div}\mathbf{u} = 0$, i.e. at constant volume, and therefore is equal to C_v/T.

The resulting equation of thermal conduction is

$$C_v \frac{\partial T}{\partial t} + \frac{C_p - C_v}{\alpha} \frac{\partial}{\partial t} \operatorname{div} \mathbf{u} = \kappa \triangle T. \qquad (32.2)$$

In order to obtain a complete system of equations, it is necessary to add an equation describing the deformation of a non-uniformly heated body. This is the equilibrium equation (7.8):

$$2(1-\sigma) \operatorname{\mathbf{grad}} \operatorname{div} \mathbf{u} - (1-2\sigma) \operatorname{\mathbf{curl}} \operatorname{\mathbf{curl}} \mathbf{u} = \tfrac{2}{3}\alpha(1+\sigma) \operatorname{\mathbf{grad}} T. \qquad (32.3)$$

From equation (32.3) we can in principle determine the deformation of the body for any given temperature distribution. Substituting the expression for div **u** thus obtained in equation (32.2), we derive an equation giving the temperature distribution, in which the only unknown function is $T(x, y, z, t)$.

For example, let us consider thermal conduction in an infinite solid in which the temperature distribution satisfies only one condition: at infinity, the temperature tends to a constant value T_0, and there is no deformation. In such a case equation (32.3) leads to the following relation between div **u** and T (see §7, Problem 8):

$$\operatorname{div} \mathbf{u} = \frac{1+\sigma}{3(1-\sigma)} \alpha (T - T_0).$$

Substituting this expression in (32.2), we obtain

$$\frac{(1+\sigma)C_p + 2(1-2\sigma)C_v}{3(1-\sigma)} \frac{\partial T}{\partial t} = \kappa \triangle T, \qquad (32.4)$$

which is the ordinary equation of thermal conduction.

An equation of this type also describes the temperature distribution along a thin straight rod, if one (or both) of its ends is free. The temperature may be assumed constant over any transverse cross-section, so that T is a function only of the co-ordinate x along the rod and of the time. The thermal expansion of such a rod causes a change in its length, but no departure from straightness and no internal stresses. Hence it is clear that the derivative $\partial S/\partial t$ in the general equation (32.1) must be taken at constant pressure and, since $(\partial S/\partial t)_p = C_p/T$, the temperature distribution will satisfy the one-dimensional thermal conduction equation $C_p \partial T/\partial t = \kappa \partial^2 T/\partial x^2$.

It should be mentioned, however, that the temperature distribution in a solid can in practice always be determined, with sufficient accuracy, by a simple thermal conduction equation. The reason is that the second term on the left-hand side of equation (32.2) is a correction of order $(C_p - C_v)/C_v$ relative to the first term. In solids, however, the difference between the two specific heats is usually very small, and if it is neglected the equation of thermal conduction in solids can always be written

$$\partial T/\partial t = \chi \triangle T, \qquad (32.5)$$

where χ is the thermometric conductivity, defined as the ratio of the thermal conductivity κ to some mean specific heat per unit volume C.

§33. Thermal conduction in crystals

In an anisotropic body, the direction of the heat flux $\mathbf{q}$ is not in general that of the temperature gradient. Hence, instead of the formula

$$\mathbf{q} = -\kappa \,\mathbf{grad}\, T$$

relating $\mathbf{q}$ to the temperature gradient, we have in a crystal the more general relation

$$q_i = -\kappa_{ik}\partial T/\partial x_k. \tag{33.1}$$

The tensor κ_{ik}, of rank two, is called the *thermal conductivity tensor* of the crystal. In accordance with (33.1), the equation of thermal conduction (32.5) has also a more general form,

$$C\frac{\partial T}{\partial t} = \kappa_{ik}\frac{\partial^2 T}{\partial x_i \partial x_k}. \tag{33.2}$$

A general theorem can be stated: the thermal conductivity tensor is symmetrical, i.e.

$$\kappa_{ik} = \kappa_{ki}. \tag{33.3}$$

This relation, which we shall now prove, is a consequence of the symmetry of the kinetic coefficients.†

The rate of increase of the total entropy of the body by irreversible processes of thermal conduction is

$$\dot{S}_{\text{tot}} = -\int \frac{\operatorname{div}\mathbf{q}}{T}\,\mathrm{d}V = -\int \operatorname{div}\frac{\mathbf{q}}{T}\,\mathrm{d}V + \int \mathbf{q}\cdot\mathbf{grad}\frac{1}{T}\,\mathrm{d}V.$$

The first integral, on being transformed into a surface integral, is seen to be zero. Thus

$$\dot{S}_{\text{tot}} = \int \mathbf{q}\cdot\mathbf{grad}\frac{1}{T}\,\mathrm{d}V = -\int \frac{\mathbf{q}\cdot\mathbf{grad}\, T}{T^2}\,\mathrm{d}V,$$

or

$$\dot{S}_{\text{tot}} = -\int \frac{1}{T^2} q_i \frac{\partial T}{\partial x_i}\,\mathrm{d}V. \tag{33.4}$$

In accordance with the general definition of the kinetic coefficients,‡ we

† See *Statistical Physics*, §122.
‡ We here use the definition given in *Fluid Mechanics*, §58

can deduce from (33.4) that in the case considered the coefficients $T^2\kappa_{ik}$ in

$$q_i = -T^2\kappa_{ik}\left(\frac{1}{T^2}\frac{\partial T}{\partial x_k}\right)$$

are kinetic coefficients. Hence the result (33.3) follows immediately from the symmetry of the kinetic coefficients.

The quadratic form

$$-q_i\frac{\partial T}{\partial x_i} = \kappa_{ik}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k}$$

must be positive, since the time derivative (33.4) of the entropy must be positive. The condition for a quadratic form to be positive is that the eigenvalues of the matrix of its coefficients are positive. Hence all the principal values of the thermal conductivity tensor κ_{ik} are always positive, this is evident also from simple considerations regarding the direction of the heat flux.

The number of independent components of the tensor κ_{ik} depends on the symmetry of the crystal. Since the tensor κ_{ik} is symmetrical, this number is evidently the same as the number for the thermal expansion tensor (§10), which is also a symmetrical tensor of rank two.

§34. Viscosity of solids

In discussing motion in elastic bodies, we have so far assumed that the deformation is reversible. In reality, this process is thermodynamically reversible only if it occurs with infinitesimal speed, so that thermodynamic equilibrium is established in the body at every instant. An actual motion, however, has finite velocities; the body is not in equilibrium at every instant, and therefore processes will take place in it which tend to return it to equilibrium. The existence of these processes has the result that the motion is irreversible, and, in particular, mechanical energy† is dissipated, ultimately into heat.

The dissipation of energy occurs by two means. Firstly, when the temperature at different points in the body is different, irreversible processes of thermal conduction take place in it. Secondly, if any internal motion occurs in the body, there are irreversible processes arising from the finite velocity of that motion. This means of energy dissipation may be referred to, as in fluids, as internal friction or *viscosity*.

In most cases the velocity of macroscopic motions in the body is so small that the energy dissipation is not considerable. Such "almost irreversible" processes can be described by means of what is called the *dissipative function*.‡

† By *mechanical energy* we here mean the sum of the kinetic energy of the macroscopic motion in the elastic body and its (elastic) potential energy arising from the deformation.

‡ See *Statistical Physics*, §121.

If we have a mechanical system whose motion involves the dissipation of energy, this motion can be described by the ordinary equations of motion, with the forces acting on the system augmented by the *dissipative forces* or *frictional forces*, which are linear functions of the velocities. These forces can be written as the velocity derivatives of a certain quadratic function Ψ of the velocities, called the dissipative function. The frictional force f_a corresponding to a generalised co-ordinate q_a of the system is then given by $f_a = -\partial\Psi/\partial\dot{q}_a$. The dissipative function Ψ is a positive quadratic form in the velocities $\dot{q}_a$. The above relation is equivalent to

$$\delta\Psi = -\sum_a f_a \delta\dot{q}_a, \tag{34.1}$$

where $\delta\Psi$ is the change in the dissipative function caused by an infinitesimal change in the velocities. It can also be shown that the dissipative function is half the decrease in the mechanical energy of the system per unit time.

It is easy to generalise equation (34.1) to the case of motion with friction in a continuous medium. The state of the system is then defined by a continuum of generalised co-ordinates. These are the displacement vector $\mathbf{u}$ at each point in the body. Accordingly, the relation (34.1) can be written in the integral form

$$\delta\int\Psi \, dV = -\int f_i \delta\dot{u}_i \, dV, \tag{34.2}$$

where f_i are the components of the dissipative force vector $\mathbf{f}$ per unit volume of the body; we write the total dissipative function for the body as $\int\Psi \, dV$, where Ψ is the dissipative function per unit volume.

Let us now determine the general form of the dissipative function Ψ for deformed bodies. The function Ψ, which describes the internal friction, must be zero if there is no internal friction, and in particular if the body executes only a general translatory or rotary motion. In other words, the dissipative function must be zero if $\dot{\mathbf{u}} = $ constant or $\dot{\mathbf{u}} = \mathbf{\Omega}\times\mathbf{r}$. This means that it must depend not on the velocity itself but on its gradient, and can contain only such combinations of the derivatives as vanish when $\dot{\mathbf{u}} = \mathbf{\Omega}\times\mathbf{r}$. These are the sums

$$\frac{\partial\dot{u}_i}{\partial x_k} + \frac{\partial\dot{u}_k}{\partial x_i},$$

i.e. the time derivatives $\dot{u}_{ik}$ of the components of the strain tensor.† Thus the dissipative function must be a quadratic function of $\dot{u}_{ik}$. The most general form of such a function is

$$\Psi = \tfrac{1}{2}\eta_{iklm}\dot{u}_{ik}\dot{u}_{lm}. \tag{34.3}$$

† Cf. the entirely analogous arguments on the viscosity of fluids in *Fluid Mechanics*, §15.

The tensor η_{iklm}, of rank four, may be called the *viscosity tensor*. It has the following evident symmetry properties:

$$\eta_{iklm} = \eta_{lmik} = \eta_{kilm} = \eta_{ikml}. \tag{34.4}$$

The expression (34.3) is exactly analogous to the expression (10.1) for the free energy of a crystal: the elastic modulus tensor is replaced by the tensor η_{iklm}, and u_{ik} by $\dot{u}_{ik}$. Hence the results obtained in §10 for the tensor λ_{iklm} in crystals of various symmetries are wholly valid for the tensor η_{iklm} also.

In particular, the tensor η_{iklm} in an isotropic body has only two independent components, and Ψ can be written in a form analogous to the expression (4.3) for the elastic energy of an isotropic body:

$$\Psi = \eta(\dot{u}_{ik} - \tfrac{1}{3}\delta_{ik}\dot{u}_{ll})^2 + \tfrac{1}{2}\zeta\dot{u}_{ll}^2, \tag{34.5}$$

where η and ζ are the two coefficients of viscosity. Since Ψ is a positive function, the coefficients η and ζ must be positive.

The relation (34.2) is entirely analogous to that for the elastic free energy, $\delta\int F\,dV = -\int F_i\delta u_i\,dV$, where $F_i = \partial\sigma_{ik}/\partial x_k$ is the force per unit volume. Hence the expression for the dissipative force f_i in terms of the tensor $\dot{u}_{ik}$ can be written down at once by analogy with the expression for F_i in terms of u_{ik}. We have

$$f_i = \partial\sigma'_{ik}/\partial x_k, \tag{34.6}$$

where the dissipative stress tensor σ'_{ik} is defined by

$$\sigma'_{ik} = \partial\Psi/\partial\dot{u}_{ik} = \eta_{iklm}\dot{u}_{lm}. \tag{34.7}$$

The viscosity can therefore be taken into account in the equations of motion by simply replacing the stress tensor σ_{ik} in those equations by the sum $\sigma_{ik} + \sigma'_{ik}$.

In an isotropic body,

$$\sigma'_{ik} = 2\eta(\dot{u}_{ik} - \tfrac{1}{3}\delta_{ik}\dot{u}_{ll}) + \zeta\dot{u}_{ll}\delta_{ik}. \tag{34.8}$$

This expression is, as we should expect, formally identical with that for the viscosity stress tensor in a fluid.

§35. The absorption of sound in solids

The absorption of sound in solids can be calculated in a manner entirely analogous to that used for fluids.† Here we shall give the calculations for an isotropic body. The thermal-conduction part of the energy dissipation $\dot{E}_{\text{mech}}$ is given by the integral $-(\kappa/T)\int(\mathbf{grad}\,T)^2\,dV$. On account of viscosity, an amount of energy 2Ψ is dissipated per unit time and volume, so that the total viscosity part of $\dot{E}_{\text{mech}}$ is $-2\int\Psi\,dV$. Using the expression (34.5), we

† See *Fluid Mechanics*, §77.

therefore have

$$\dot{E}_{\text{mech.}} = -\frac{\kappa}{T}\int (\mathbf{grad}\, T)^2\, \mathrm{d}V - 2\eta\int (\dot{u}_{ik} - \tfrac{1}{3}\delta_{ik}\dot{u}_{ll})^2\, \mathrm{d}V - \zeta\int \dot{u}_{ll}^2\, \mathrm{d}V. \quad (35.1)$$

To calculate the temperature gradient, we use the fact that sound oscillations are adiabatic in the first approximation. Using the expression (6.4) for the entropy, we can write the adiabatic condition as $S_0(T)+K\alpha u_{ll} = S_0(T_0)$, where T_0 is the temperature in the undeformed state. Expanding the difference $S_0(T) - S_0(T_0)$ in powers of $T - T_0$, we have as far as the first-order terms $S_0(T) - S_0(T_0) = (T - T_0)\,\partial S_0/\partial T_0 = C_v(T-T_0)/T_0$. The derivative of the entropy is taken for $u_{ll} = 0$, i.e. at constant volume. Thus

$$T - T_0 = -T\alpha K u_{ll}/C_v.$$

Using also the relations $K = K_{\text{iso}} = C_v K_{\text{ad}}/C_p$ and $K_{\text{ad}}/\rho = c_l^2 - 4c_t^2/3$, we can rewrite this result as

$$T - T_0 = -\frac{T\alpha\rho(c_l^2 - 4c_t^2/3)}{C_p} u_{ll}. \quad (35.2)$$

Let us first consider the absorption of transverse sound waves. The thermal conduction cannot result in the absorption of these waves (in the approximation considered). For, in a transverse wave, we have $u_{ll} = 0$, and therefore the temperature in it is constant, by (35.2). Let the wave be propagated along the x-axis; then

$$u_x = 0, \quad u_y = u_{0y}\cos(kx - \omega t), \quad u_z = u_{0z}\cos(kx - \omega t),$$

and the only non-zero components of the deformation tensor are $u_{xy} = -\frac{1}{2}ku_{0y}\sin(kx - \omega t)$, $u_{xz} = -\frac{1}{2}ku_{0z}\sin(kx - \omega t)$.

We shall consider the energy dissipation per unit volume of the body; the (time) average value of this quantity is, from (35.1),

$$\bar{\dot{E}}_{\text{mech}} = -\tfrac{1}{2}\eta\omega^4(u_{0y}^2 + u_{0z}^2)/c_t^2,$$

where we have put $k = \omega/c_t$. The total mean energy of the wave is twice the mean kinetic energy, i.e.

$$\bar{E} = \rho\int \bar{\dot{u}^2}\, \mathrm{d}V;$$

for unit volume we have

$$\bar{E} = \tfrac{1}{2}\rho\omega^2(u_{0y}^2 + u_{0z}^2).$$

The sound absorption coefficient is defined as the ratio of the mean energy dissipation to twice the mean energy flux in the wave; this quantity gives the manner of variation of the wave amplitude with distance. The amplitude

decreases as $e^{-\gamma x}$. Thus we find the following expression for the absorption coefficient for transverse waves:

$$\gamma_t = \tfrac{1}{2}|\bar{\dot{E}}_{\text{mech}}|/c_t\bar{E} = \eta\omega^2/2\rho c_t^3. \tag{35.3}$$

In a longitudinal sound wave $u_x = u_0 \cos(kx - \omega t)$, $u_y = u_z = 0$. A similar calculation, using formulae (35.1) and (35.2), gives

$$\gamma_l = \frac{\omega^2}{2\rho c_l^3}\left[\left(\frac{4}{3}\eta + \zeta\right) + \frac{\kappa T\alpha^2\rho^2 c_l^2}{C_p^2}\left(1 - \frac{4c_t^2}{3c_l^2}\right)^2\right]. \tag{35.4}$$

These formulae relate, strictly speaking, only to a completely isotropic and amorphous body. They give, however, the correct order of magnitude for the absorption of sound in anisotropic single crystals also.

The absorption of sound in polycrystalline bodies exhibits peculiar properties. If the wavelength λ of the sound is small in comparison with the dimensions a of the individual crystallites, then the sound is absorbed in each crystallite in the same way as in a large crystal, and the absorption coefficient is proportional to ω^2.

If $\lambda \gg a$, however, the nature of the absorption is different. In such a wave we can assume that each crystallite is subject to a uniformly distributed pressure. However, since the crystallites are anisotropic, and so are the boundary conditions at their surfaces of contact, the resulting deformation is not uniform. It varies considerably (by an amount of the same order as itself) over the dimension of a crystallite, and not over one wavelength as in a homogeneous body. When sound is absorbed, the rates of change of the deformation ($\dot{u}_{ik}$) and the temperature gradients are of importance. Of these, the former are still of the usual order of magnitude. The temperature gradients within each crystallite are anomalously large, however. Hence the absorption due to thermal conduction will be large compared with that due to viscosity, and only the former need be calculated.

Let us consider two limiting cases. The time during which the temperature is equalised by thermal conduction over distances $\sim a$ (the relaxation time for thermal conduction) is of the order of a^2/χ. Let us first assume that $\omega \ll \chi/a^2$. This means that the relaxation time is small compared with the period of the oscillations in the wave, and so thermal equilibrium is nearly established in each crystallite; in this case we have almost isothermal oscillations.

Let T' be the temperature difference in a crystallite, and T_0' the corresponding difference in an adiabatic process. The heat transferred by thermal conduction per unit volume is $-\text{div}\,\mathbf{q} = \kappa\triangle T' \sim \kappa T'/a^2$. The amount of heat evolved in the deformation is of the order of $\dot{T}_0'C \sim \omega T_0'C$, where C is the specific heat. Equating the two, we obtain $T' \sim T_0'\omega a^2/\chi$. The temperature varies by an amount of the order of T' over the dimension of the crystallite, and so its gradient is of magnitude $\sim T'/a$. Finally, T_0' is

found from (35.2), with $u_{ll} \sim ku \sim \omega u/c$ (u being the amplitude of the displacement vector):

$$T_0' \sim T\alpha\rho c\omega u/C; \tag{35.5}$$

in obtaining orders of magnitude, we naturally neglect the difference between the various velocities of sound. Using these results, we can calculate the energy dissipated per unit volume:

$$\bar{\dot{E}}_{\mathrm{max}} \sim \frac{\kappa}{T}(\mathbf{grad}\, T')^2 \sim \frac{\kappa}{T}\left(\frac{T'}{a}\right)^2.$$

Dividing this by the energy flux $c\bar{E} \sim c\rho\omega^2u^2$, we find the damping coefficient to be

$$\gamma \sim T\alpha^2\rho c a^2\omega^2/\chi C \text{ for } \omega \ll \chi/a^2 \tag{35.6}$$

(C. ZENER 1938). Comparing this expression with the general expressions (35.3) and (35.4), we can say that, in the case considered, the absorption of sound by a polycrystalline body is the same as if it had a viscosity

$$\eta \sim T\alpha^2\rho^2c^4a^2/\chi C,$$

which is much larger than the actual viscosity of the component crystallites.

Next, let us consider the opposite limiting case, where $\omega \gg \chi/a^2$. In other words, the relaxation time is large compared with the period of oscillations in the wave, and no noticeable equalisation of the temperature differences due to the deformation can occur in one period. It would be incorrect, however, to suppose that the temperature gradients which determine the absorption of sound are of the order of T_0'/a. This assumption would take into account only thermal conduction in each crystallite, whereas heat exchange between neighbouring crystallites must be of importance in the case in question (M. A. ISAKOVICH 1948). If the crystallites were thermally insulated the temperature differences occurring at their boundaries would be of the same order T_0' as those within each individual crystallite. In reality, however, the boundary conditions require the continuity of the temperature across the surface separating two crystallites. We therefore have "temperature waves" propagated away from the boundary into the crystallite; these are damped at a distance† $\delta \sim \sqrt{(\chi/\omega)}$. In the case under consideration $\delta \ll a$, i.e. the main temperature gradient is of the order of T_0'/δ and occurs over distances small compared with the total dimension of a crystallite. The corresponding fraction of the volume of the crystallite is $\sim a^2\delta$; taking the ratio

† It may be recalled that, if a thermally conducting medium is bounded by the plane $x = 0$, at which the excess temperature varies periodically according to $T' = T_0' e^{-i\omega t}$, then the temperature distribution in the medium is given by the "temperature wave" $T' = T_0' e^{-i\omega t} e^{-(1+i)x\sqrt{(\omega/2\chi)}}$; see *Fluid Mechanics*, §52.

of this to the total volume $\sim a^3$, we find the mean energy dissipation

$$\bar{\dot{E}}_{\text{mech}} \sim \frac{\kappa}{T}\left(\frac{T_0'}{\delta}\right)^2 \frac{a^2\delta}{a^3} \approx \frac{\kappa T_0'^2}{Ta\delta}.$$

Substituting for T_0' the expression (35.5) and dividing by $c\bar{E} \sim c\rho\omega^2 u^2$, we obtain the required absorption coefficient:

$$\gamma \sim T\alpha^2 \rho c\sqrt{(\chi\omega)}/aC \text{ for } \omega \gg \chi/a^2. \tag{35.7}$$

It is proportional to the square root of the frequency.†

Thus the sound absorption coefficient in a polycrystalline body varies as ω^2 at very low frequencies ($\omega \ll \chi/a^2$); for $\chi/a^2 \ll \omega \ll c/a$ it varies as $\sqrt{\omega}$, and for $\omega \gg c/a$ it again varies as ω^2.

Similar considerations hold for the damping of transverse waves in thin rods and plates (C. ZENER 1938). If h is the thickness of the rod or plate, then for $\lambda \gg h$ the transverse temperature gradient is important, and the damping is mainly due to thermal conduction (see the Problems). If also $\omega \ll \chi/h^2$, the oscillations may be regarded as isothermal, and therefore, in determining (for example) the characteristic frequencies of vibrations of the rod or plate, the isothermal values of the moduli of elasticity must be used.

PROBLEMS

PROBLEM 1. Determine the damping coefficient for longitudinal vibrations of a rod.

SOLUTION. The damping coefficient for the vibrations is defined as $\beta = |\bar{\dot{E}}_{\text{mech}}|/2\bar{E}$; the amplitude of the vibrations diminishes with time as $e^{-\beta t}$.

In a longitudinal wave, any short section of the rod is subject to simple extension or compression; the components of the strain tensor are $u_{zz} = \partial u_z/\partial z$, $u_{xx} = u_{yy} = -\sigma_{\text{ad}}\partial u_z/\partial z$. We put $u_z = u_0 \cos kz \cos \omega t$, where $k = \omega/\sqrt{(E_{\text{ad}}/\rho)}$. Calculations similar to those given above lead to the following expression for the damping coefficient:

$$\beta = \frac{\omega^2}{2\rho}\left\{\frac{\eta}{3}\frac{3c_l^2-4c_t^2}{(c_l^2-c_t^2)c_t^2} + \frac{\zeta c_t^2}{(c_l^2-c_t^2)(3c_l^2-4c_t^2)} + \frac{\kappa T\alpha^2\rho^2}{9C_p^2}\right\}.$$

Here we have written E_{ad} and σ_{ad} in terms of the velocities c_l, c_t by means of formulae (22.4).

PROBLEM 2. The same as Problem 1, but for longitudinal oscillations of a plate.

SOLUTION. For waves whose direction of oscillation is parallel to that of their propagation (the x-axis, say) the non-zero components of the strain tensor are

$$u_{xx} = \partial u_x/\partial x, \qquad u_{zz} = -[\sigma_{\text{ad}}/(1-\sigma_{\text{ad}})]\partial u_x/\partial x;$$

see (13.1). The velocity of propagation of these waves is $\sqrt{[E_{\text{ad}}/\rho(1-\sigma_{\text{ad}}^2)]}$. A calculation gives

$$\beta = \frac{\omega^2}{2\rho}\left\{\frac{\eta}{3}\frac{3c_l^4+4c_t^4-6c_l^2c_t^2}{c_t^2c_l^2(c_l^2-c_t^2)} + \frac{\zeta c_t^2}{c_l^2(c_l^2-c_t^2)} + \frac{\kappa T\alpha^2\rho^2(1+\sigma_{\text{ad}})^2}{9C_p^2}\right\}.$$

† The same frequency dependence is found for the absorption of sound propagated in a fluid near a solid wall (in a pipe, for instance); see *Fluid Mechanics*, §77, Problems.

For waves whose direction of oscillation is perpendicular to the direction of propagation, $u_{ll} = 0$, and the damping is caused only by the viscosity η. In this case the damping coefficient is $\beta = \eta\omega^2/2\rho c_t^2$. This applies also to the damping of torsional vibrations of rods.

PROBLEM 3. Determine the damping coefficient for transverse vibrations of a rod (with frequencies such that $\omega \gg \chi/h^2$, where h is the thickness of the rod).

SOLUTION. The damping is due mainly to thermal conduction. According to §17, we have for each volume element in the rod $u_{zz} = x/R$, $u_{xx} = u_{yy} = -\sigma_{\text{ad}}x/R$ (for bending in the xz-plane); for $\omega \gg \chi/h^2$, the vibrations are adiabatic. For small deflections the radius of curvature $R = 1/X''$, so that $u_{ii} = (1-2\sigma_{\text{ad}})xX''$, the prime denoting differentiation with respect to z. The temperature varies most rapidly across the rod, and so $(\textbf{grad}\ T)^2 \approx (\partial T/\partial x)^2$. Using (35.1) and (35.2), we obtain for the total mean energy dissipation in the rod $-(\kappa T\alpha^2 E_{\text{ad}}{}^2 S/9C_p{}^2)\int \overline{X''^2}\,dz$, where S is the cross-sectional area of the rod. The mean total energy is twice the potential energy $E_{\text{ad}}I_y\int \overline{X''^2}\,dz$. The damping coefficient is

$$\beta = \kappa T\alpha^2 S E_{\text{ad}}/18 I_y C_p{}^2.$$

PROBLEM 4. The same as Problem 3, but for transverse vibrations of a plate.

SOLUTION. According to (11.4), we have for any volume element in the plate

$$u_{ii} = -\frac{1-2\sigma_{\text{ad}}}{1-\sigma_{\text{ad}}}z\frac{\partial^2\zeta}{\partial x^2}$$

for bending in the xz-plane. The energy dissipation is found from formulae (35.1) and (35.2) and the mean total energy is twice the expression (11.6). The damping coefficient is

$$\beta = \frac{2\kappa T\alpha^2 E_{\text{ad}}}{3C_p{}^2h^2}\cdot\frac{1+\sigma_{\text{ad}}}{1-\sigma_{\text{ad}}} = \frac{2\kappa T\alpha^2\rho}{3C_p{}^2h^2}\cdot\frac{(3c_l{}^2-4c_t{}^2)^2c_t{}^2}{(c_l{}^2-c_t{}^2)c_l{}^2}.$$

PROBLEM 5. Determine the change in the characteristic frequencies of transverse vibrations of a rod due to the fact that the vibrations are not adiabatic. The rod is in the form of a long plate of thickness h. The surface of the rod is supposed thermally insulated.

SOLUTION. Let $T_{\text{ad}}(x, t)$ be the temperature distribution in the rod for adiabatic vibrations, and $T(x, t)$ the actual temperature distribution; x is a co-ordinate across the thickness of the rod, and the temperature variation in the yz-plane is neglected. Since, for $T = T_{\text{ad}}$, there is no heat exchange between various parts of the body, it is clear that the thermal conduction equation must be

$$\frac{\partial}{\partial t}(T-T_{\text{ad}}) = \chi\frac{\partial^2 T}{\partial x^2}.$$

For periodic vibrations of frequency ω, the differences $\tau_{\text{ad}} = T_{\text{ad}}-T_0$, $\tau = T-T_0$ from the equilibrium temperature T_0 are proportional to $e^{-i\omega t}$, and we have $\tau''+i\omega\tau/\chi = i\omega\tau_{\text{ad}}/\chi$, the prime denoting differentiation with respect to x. Since, by (35.2), τ_{ad} is proportional to u_{ll}, and the components u_{ik} are proportional to x (see §17), it follows that $\tau_{\text{ad}} = Ax$, where A is a constant which need not be calculated, since it does not appear in the final result. The solution of the equation $\tau''+i\omega\tau/\chi = i\omega Ax/\chi$, with the boundary condition $\tau' = 0$ for $x = \pm\frac{1}{2}h$ (the surface of the rod being insulated), is

$$\tau = A\left(x - \frac{\sin kx}{k\cos\frac{1}{2}kh}\right), \qquad k = (1+i)\sqrt{(\omega/2\chi)}.$$

The moment M_y of the internal stress forces in a rod bent in the xz-plane is composed of the isothermal part $M_{y,\text{iso}}$ (i.e. the value for isothermal bending) and the part due to the

non-uniform heating of the rod. If $M_{y,\mathrm{ad}}$ is the moment in adiabatic bending, the second part of the moment is reduced from $M_{y,\mathrm{ad}} - M_{y,\mathrm{iso}}$ in the ratio

$$1+f(\omega) = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} z\tau \,\mathrm{d}z \Big/ \int_{-\frac{1}{2}h}^{\frac{1}{2}h} z\tau_{\mathrm{ad}} \,\mathrm{d}z.$$

Defining the YOUNG's modulus E_ω for any frequency ω as the coefficient of proportionality between M_y and I_y/R (see (17.8)), and noticing that $E_{\mathrm{ad}} - E = E^2T\alpha^2/9C_p$ (see (6.8); E is the isothermal YOUNG's modulus), we can put

$$E_\omega = E + [1+f(\omega)]E^2T\alpha^2/9C_p.$$

A calculation shows that $f(\omega) = (24/k^3h^3)(\frac{1}{2}kh - \tan\frac{1}{2}kh)$. For $\omega \to \infty$ we obtain $f \to 1$, which is correct, since $E_\infty = E_{\mathrm{ad}}$, and for $\omega \to 0$, $f \to 0$ and $E_\omega \to E$.

The frequencies of the characteristic vibrations are proportional to the square root of the YOUNG's modulus (see §25, Problems 4–6). Hence

$$\omega = \omega_0\left[1 + f(\omega_0)\frac{ET\alpha^2}{18C_p}\right],$$

where ω_0 are the characteristic frequencies for adiabatic vibrations. This value of ω is complex. Separating the real and imaginary parts $(\omega = \omega' + i\beta)$, we find the characteristic frequencies

$$\omega' = \omega_0\left[1 - \frac{ET\alpha^2}{3C_p}\cdot\frac{1}{\xi^3}\cdot\frac{\sinh\xi - \sin\xi}{\cosh\xi + \cos\xi}\right]$$

and the damping coefficient

$$\beta = \frac{2ET\alpha^2\chi}{3C_ph^2}\left[1 - \frac{1}{\xi}\cdot\frac{\sinh\xi + \sin\xi}{\cosh\xi + \cos\xi}\right],$$

where $\xi = h\sqrt{(\omega_0/2\chi)}$.

For large ξ the frequency ω tends to ω_0, as it should, and the damping coefficient to $2ET\alpha^2\chi/3C_ph^2$, in accordance with the result of Problem 3.

Small values of ξ correspond to almost isothermal conditions; in this case

$$\omega \cong \omega_0\left(1 - \frac{ET\alpha^2}{18C_p}\right) \approx \omega_0\sqrt{(E/E_{\mathrm{ad}})},$$

and the damping coefficient $\beta = ET\alpha^2h^2\omega_0^2/180C_p\chi$.

§36. Highly viscous fluids

For typical fluids, the Navier–Stokes equations are valid if the periods of the motion are large compared with times characterising the molecules. This, however, is not true for very viscous fluids. In such fluids, the usual equations of fluid mechanics become invalid for much larger periods of the motion. There are viscous fluids which, during short intervals of time (though these are long compared with molecular times), behave as solids (for instance, glycerine and resin). Amorphous solids (for instance, glass) may be regarded as a limiting case of such fluids having a very large viscosity.

The properties of these fluids can be described by the following method, due to MAXWELL. They are elastically deformed during short intervals of time.

When the deformation ceases, shear stresses remain in them, although these are damped in the course of time, so that after a sufficiently long time almost no internal stress remains in the fluid. Let τ be of the order of the time during which the stresses are damped (sometimes called the *Maxwellian relaxation time*). Let us suppose that the fluid is subjected to some variable external forces, which vary periodically in time with frequency ω. If the period $1/\omega$ is large compared with the relaxation time τ, i.e. $\omega\tau \ll 1$, the fluid under consideration will behave as an ordinary viscous fluid. If, however, the frequency ω is sufficiently large (so that $\omega\tau \gg 1$), the fluid will behave as an amorphous solid.

In accordance with these "intermediate" properties, the fluids in question can be characterised by both a viscosity coefficient η and a modulus of rigidity μ. It is easy to obtain a relation between the orders of magnitude of η, μ and the relaxation time τ. When periodic forces of sufficiently small frequency act, and so the fluid behaves like an ordinary fluid, the stress tensor is given by the usual expression for viscosity stresses in a fluid, i.e.

$$\sigma_{ik} = 2\eta\dot{u}_{ik} = -2i\omega\eta u_{ik}.$$

In the opposite limit of large frequencies, the fluid behaves like a solid, and the internal stresses must be given by the formulae of the theory of elasticity, i.e. $\sigma_{ik} = 2\mu u_{ik}$; we are speaking of pure shear deformations, i.e. we assume that $u_{ii} = \sigma_{ii} = 0$. For frequencies $\omega \sim 1/\tau$, the stresses given by these two expressions must be of the same order of magnitude. Thus $\eta u/\lambda\tau \sim \mu u/\lambda$, whence

$$\eta \sim \tau\mu. \tag{36.1}$$

This is the required relation.

Finally, let us derive the equation of motion which qualitatively describes the behaviour of these fluids. To do so, we make a very simple assumption concerning the damping of the internal stresses (when motion ceases): namely, that they are damped exponentially, i.e. $\mathrm{d}\sigma_{ik}/\mathrm{d}t = -\sigma_{ik}/\tau$. In a solid, however, we have $\sigma_{ik} = 2\mu u_{ik}$, and so $\mathrm{d}\sigma_{ik}/\mathrm{d}t = 2\mu \mathrm{d}u_{ik}/\mathrm{d}t$. It is easy to see that the equation

$$\frac{\mathrm{d}\sigma_{ik}}{\mathrm{d}t} + \frac{\sigma_{ik}}{\tau} = 2\mu\frac{\mathrm{d}u_{ik}}{\mathrm{d}t} \tag{36.2}$$

gives the correct result in both limiting cases of slow and rapid motions, and may therefore serve as an interpolatory equation for intermediate cases.

For example, in periodic motion, where u_{ik} and σ_{ik} depend on the time through a factor $e^{-i\omega t}$, we have from (36.2) $-i\omega\sigma_{ik} + \sigma_{ik}/\tau = -2i\omega\mu u_{ik}$, whence

$$\sigma_{ik} = \frac{2\mu u_{ik}}{1 + i/\omega\tau}. \tag{36.3}$$

For $\omega\tau \gg 1$, this formula gives $\sigma_{ik} = 2\mu u_{ik}$, i.e. the usual expression for solid bodies, while for $\omega\tau \ll 1$ we have $\sigma_{ik} = -2i\omega\mu\tau u_{ik} = 2\mu\tau\dot{u}_{ik}$, the usual expression for a fluid of viscosity $\mu\tau$.

INDEX